9.8 Jordan-Brouwer Separation Theorems

Definition 9.8.1 *Suppose $A \subset X$. We say that A* separates *X if $X \smallsetminus A$ is disconnected (i.e, not path connected), or equivalently if $\tilde{H}_*(X \smallsetminus A) \neq 0$.*

Terminology: If B is homeomorphic to D^k then B is called a *k-cell*.

Theorem 9.8.2 *Let $B \subset S^n$ be a k-cell. Then $S^n \smallsetminus B$ is acyclic. (i.e. $\tilde{H}(S^n \smallsetminus B) = 0 \ \forall \ q$.) In particular, B does not separate S^n.*

Remark 9.8.3 *$B \simeq *$ and $S^n \smallsetminus \{*\} = \mathbb{R}^n$, but in general $A \simeq B$ does not imply that $X \smallsetminus A \simeq X \smallsetminus B$.*

Proof: By induction on k.
$k = 0$ is trivial since then $B = *$ and $S^n \smallsetminus \{*\} = \mathbb{R}^n$.
Suppose that the theorem is true for $(k-1)$-cells.
Let $h : I^k \to B$ be a homeomorphism.
Write $B = B_1 \cup B_2$ where $B_1 := h(I^{k-1} \times [0, 1/2])$ and $B_2 := h(I^{k-1} \times [1/2, 1])$.
Let $C = B_1 \cap B_2$; a $(k-1)$-cell.
Let $i : (S^n \smallsetminus B) \to S^n \smallsetminus B_1$, $j : (S^n \smallsetminus B) \to (S^n \smallsetminus B_2)$.
Suppose $0 \neq \alpha \in \tilde{H}_p(S^n \smallsetminus B)$.

Lemma 9.8.4 *Either $i_*(\alpha) \neq 0$ or $j_*(\alpha) \neq 0$.*

Proof: $S^n \smallsetminus B_1$ and $S^n \smallsetminus B_2$ are open so they have a Mayer-Vietoris sequence.
$(S^n \smallsetminus B_1) \cap (S^n \smallsetminus B_2) = S^n \smallsetminus B$ $\quad (S^n \smallsetminus B_1) \cup (S^n \smallsetminus B_2) = S^n \smallsetminus (B_1 \cap B_2) = S^n \smallsetminus C$.

$$\begin{array}{ccccc} \tilde{H}_{p+1}(S^n \smallsetminus C) & \xrightarrow{\ \Delta\ } & \tilde{H}_p(S^n \smallsetminus B) & \xrightarrow{(i_*, j_*)} & \tilde{H}_p(S^n \smallsetminus B_1) \oplus \tilde{H}_p(S^n \smallsetminus B_2) \\ \text{(by hypothesis)} \Vert & & & & \\ 0 & & & & \end{array}$$

so either $i_*(\alpha) \neq 0$ or $j_*(\alpha) \neq 0$. ◻

Proof of Theorem (cont.): By the lemma, continuing to subdivide we obtain a nested decreasing sequence of closed intervals I_n s.t. if we let $j_m : (S^n \smallsetminus B) \hookrightarrow (S^n \smallsetminus Q_m)$, where $Q_m := h(I^{k-1} \times I_m)$, then $j_{m*}\alpha \neq 0$.
By the Cantor Intersection Theorem, $\cap_m I_m =$ a single point $\{e\}$.

$$\begin{array}{ccccccc} H_p(S^n \setminus B) & \longrightarrow \dots \longrightarrow & H_p(S^n \setminus Q_m) & \longrightarrow \dots \longrightarrow & H_p\big(S^n \setminus h(I^{k-1} \times \{e\})\big) \\ & & & & \text{(induction)}\Big\| \\ & & & & 0 \end{array}$$

where we have used that $E := h(I^{k-1} \times \{e\})$ is a $(k-1)$-cell. Since $S^n \setminus Q_m$ is open and nested and $S^n \setminus E = \cup_{m=1}^{\infty}(S^n \setminus Q_m)$, $H_*(S^n \setminus E) = \varinjlim H_*(S^n \setminus Q_m)$.

Therefore $\alpha \mapsto 0$ in $H_*(S^n \setminus E)$ implies that $j_{m*}(\alpha) = 0$ in $H_*(S^n \setminus Q_m)$ for some m, which is a contradiction. Hence $\nexists$ nonzero $\alpha \in H_p(S^n \setminus B)$. □

Theorem 9.8.5 *Suppose* $h : S^k \hookrightarrow S^n$. *Then* $\tilde{H}_i\big(S^n \setminus h(S^k)\big) = \begin{cases} \mathbb{Z} & i = n-k-1; \\ 0 & \text{otherwise.} \end{cases}$

Proof: By induction on k.

If $k = 0$, $\tilde{H}_p\big(S^n \setminus h(S^0)\big) = \tilde{H}_p\big(S^n \setminus \{2 \text{ points}\}\big) = \tilde{H}_p\big(\mathbb{R}^n \setminus \{\text{point}\}\big) = \tilde{H}_p(S^{n-1})$.

√

Suppose that the theorem is true for $k-1$.

Let E^k_+, E^k_- be the upper and lower hemispheres of S^k. Notice that by compactness, h is a homeomorphism onto its image, so $h(E^k_+)$ and $h(E^k_-)$ are k-cells.

Also $S^n \setminus h(E^k_+)$, $S^n \setminus h(E^k_-)$ are open so Mayer-Vietoris applies.

$\big(S^n \setminus h(E^k_+)\big) \cup \big(S^n \setminus h(E^k_-)\big) = \big(S^n \setminus h(E^k_+ \cap E^k_-)\big) = \big(S^n \setminus h(S^{k-1})\big)$

$\big(S^n \setminus h(E^k_+)\big) \cap \big(S^n \setminus h(E^k_-)\big) = \big(S^n \setminus h(E^k_+ \cup E^k_-)\big) = \big(S^n \setminus h(S^k)\big)$

$$\begin{array}{c} 0 \\ \| \\ \tilde{H}_p\big(S^n \setminus h(E^k_+)\big) \oplus \tilde{H}_p\big(S^n \setminus h(E^k_-)\big) \end{array} \to \tilde{H}_p\big(S^n \setminus h(S^{k-1})\big) \xrightarrow[\cong]{\Delta} \tilde{H}_{p-1}\big(S^n \setminus h(S^k)\big)$$

$$\to \begin{array}{c} \tilde{H}_{p-1}\big(S^n \setminus h(E^k_+)\big) \oplus \tilde{H}_{p-1}\big(S^n \setminus h(E^k_-)\big) \\ \| \\ 0 \end{array}$$

□

Theorem 9.8.6 *(Jordan Curve Theorem) Suppose* $n > 0$. *Let* C *be a subset of* S^n *which is homeomorphic to* S^{n-1}. *Then* $S^n \setminus C$ *has precisely two path components and* C *is their common boundary. (Furthermore, the components are open in* S^n.*)*

Proof: By the preceding theorem, $\tilde{H}_0(S^n \setminus C) \cong \mathbb{Z}$, so $S^n \setminus C$ has two path components. Denote these components W_1 and W_2.

C is closed in S^n so $S^n \setminus C$ is open. Hence by local path connectedness of S^n, its components W_1 and W_2 are open. Thus $\overline{W_1} \subset W_2^c$.

If $x \in \partial W_1 = \overline{W_1} \setminus W_1$, then $x \notin W_2$ (since $x \in \overline{W_1} = W_2^c$) and $x \notin W_1$. So $x \in (W_1 \cup W_2)^c = C$. Hence $\partial W_1 \subset C$.

Conversely let $x \in C$.

Let U be an open neighbourhood of x. Show $U \cap \overline{W_1} \neq \emptyset$. Since U arbitrary, it will follow that x is an accumulation point of $\overline{W_1}$ so that $x \in \overline{W_1}$. But $x \in C$ so $x \notin W_1$, resulting in $x \in \overline{W_1} \setminus W_1 = \partial W_1$.

To show $U \cap \overline{W_1} \neq \emptyset$:

$U \cap C$ is homeomorphic to an open subset of S^{n-1} (since $C \cong S^{n-1}$ by hypothesis) so it contains the closure of an $(n-1)$-sphere. Let C_1 be this closure. Under the homeomorphism $C \cong S^{n-1}$, $C_1 \cong N_r[x]$ for some r and x. Thus $C_1 \subset C$ is an $(n-1)$-cell. Let $C_2 = \overline{C \setminus C_1}$. Then C_2 is also an $(n-1)$-cell (up to homeomorphism it is the closure of the complement of $N_r[x]$ in S^{n-1}) and $C_1 \cup C_2 = C$ which is closed. By Theorem 9.8.2, C_2 does not separate S^n so $\exists$ path α in $S^n \setminus C_2$ joining $p \in W_1$ to $q \in W_2$. $(\operatorname{Im}\alpha) \cap (\overline{W_1} \setminus W_1) = \alpha\big(\alpha^{-1}(\overline{W_1}) \setminus \alpha^{-1}(W_1)\big)$. If this is empty then $\alpha^{-1}(\overline{W_1}) = \alpha^{-1}(W_1)$. However the equality of these open and closed subsets of I means that either $\alpha(W_1) = \emptyset$ or $\alpha^{-1}(W_1) = I$. We know $\alpha^{-1}(W_1) \neq \emptyset$ since $0 \in \alpha^{-1}(W_1)$ (since $p = \alpha(0) \in W_1$). And $1 \notin \alpha^{-1}(W_1)$ since $q \notin W_1$. Therefore $(\operatorname{Im}\alpha) \cap (\overline{W_1} \setminus W_1) \neq \emptyset$. Thus $\exists y \in (\operatorname{Im}\alpha) \cap (\overline{W_1} \setminus W_1) \subset \partial W_1 \subset C = C_1 \cup C_2$. Since $\operatorname{Im}\alpha \subset S^n \setminus C_2$, $y \notin C_2$ so $y \in C_1 \subset U$. Hence $y \in U \cap \overline{W_1}$. $\checkmark$

So $\partial W_1 = C$. Similarly $\partial W_2 = C$, as desired. □

Corollary 9.8.7 *(Jordan Curve Theorem - standard version): Supppose $n > 1$. Let C be asubspace of $\mathbb{R}^n$ which is homeomorphic to S^{n-1}. Then $\mathbb{R}^n \setminus C$ has precisely two components (one bounded, one unbounded — known as the "inside of C" and "outside of C" respectively) and C is their common boundary.*

Proof: Include $\mathbb{R}^n$ into S^n, writing $\mathbb{R}^n = S^n = \{P\}$. Then $S^n \setminus C$ is the union of two components W_1, W_2 whose common boundary is C. One of the components, say W_1 contains P so $W_1 \setminus \{P\}$, W_2 are the components of $\mathbb{R}^n \setminus C$ and their common boundary is C. □

Theorem 9.8.8 *(Invariance of Domain): Let V be open in $\mathbb{R}^n$ and let $f : V \to \mathbb{R}^n$ be continuous and injective. Then $f(V)$ is open in $\mathbb{R}^n$ and $f : V \to f(V)$ is a homeomorphism.*

Remark 9.8.9 *Compare the inverse function theorem which asserts this under the stronger hypothesis that f is continuously differentiable with non-singular Jacobian, but also asserts differentiability of the inverse map.*

Proof:

Include $\mathbb{R}^n$ into S^n. Let U be an open subset of V. Let $y \in f(U)$. We show that $f(U)$ contains an open neighbourhood of y.

Write $y = f(x)$, Find ϵ s.t. $N_\epsilon[x] \subset U$. Set $A := N_\epsilon[x] \smallsetminus N_\epsilon(x)$. So A is homeomorphic to S^{n-1}. Since $f\big|_{N_\epsilon[x]} \subset U$ is a homeomorphism (an injective map from a compact set to a Hausdorff space), $f(A)$ is homeomorphic to S^{n-1}. Therefore $f(A)$ separates S^n into two components W_1 and W_2 which are open in S^n.

$N_\epsilon(x)$ is connected and disjoint from A, so $f\big(N_\epsilon(x)\big)$ is connected and disjoint from $f(A)$. Thus $f\big(N_\epsilon(x)\big)$ is contained entirely within either W_1 or W_2. Say $f\big(N_\epsilon(x)\big) \subset W_1$.

$$S^n \smallsetminus f(A) \smallsetminus f\big(N_\epsilon(x)\big) = S^n \smallsetminus f\big(A \cup N_\epsilon(x)\big) = S^n \smallsetminus f\big(N_\epsilon[x]\big)$$

(which the later argument will show is equal to $S^n \smallsetminus W_2^c = W_2$). Since $f\big(N_\epsilon[x]\big)$ is an n-cell, it does not disconnect S^n, i.e. $S^n \smallsetminus f\big(N_\epsilon[x]\big)$ is connected. Because $f\big(N_\epsilon[x]\big) \subset \overline{W_1} \subset W_2^c$ which is equivalent to $W_2 \subset S^n \smallsetminus f\big(N_\epsilon[x]\big)$, we get $W_2 = S^n \smallsetminus f\big(N_\epsilon[x]\big)$ (as remarked earlier), since W_2 is a path component of S^n. Hence $f\big(N_\epsilon[x]\big) = W_2^c = \overline{W_1}$. Thus $f\big(N_\epsilon(x)\big) = W_1$. (i.e. If $z \in W_1 \smallsetminus f\big(N_\epsilon(x)\big)$ then $z \in S^n \smallsetminus f(A) \smallsetminus f\big(N_\epsilon(x)\big) = S^n \smallsetminus f\big(N_\epsilon[x]\big) = S^n \smallsetminus W^c = W_2$, which contradicts $W_1 \cap W_2 = \emptyset$.)

Therefore we have shown that $\exists$ an open set W_1 s.t. $y \in W_1 \subset f(U)$ and thus $f(U)$ is open. Applying the above argument with $U := V$ gives that $f(V)$ is open. It also shows that $f : V \to f(V)$ is an open map, so it is a homeomorphism. $\square$