9.16 Hurewicz Homomorphism

Suppose $n \geq 1$ and let ι_n be a generator of $H_n(S^n)$. Define $h : \pi_n(X) \to H_n(X)$ by $h([f]) := f_*(\iota_n)$ for a representative $f : S^n \to X$. This is well defined by the homotopy axiom.

Check that h is a group homomorphism:

$[fg] = [(f \perp g) \circ \psi]$ where $\psi : S^n \to S^n \vee S^n$ pinches the equator to a point.

$H_n(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $e_1 := j_1(\iota)$, $e_2 := j_1(\iota)$, where $j_1, j_2 : S^n \to S^n \vee S^n \subset S^n \times S^n$ by $j_1(x) = (x, *)$ and $j_2(x) = (*, x)$. $\psi_*(\iota) = e_1 + e_2$. To determine $(f \perp g)_*(e_1)$ use the commutative diagram

$$\begin{tikzcd}
S^n \arrow[r, "j_1"] \arrow[dr, "f \simeq f \cdot *"'] & S^n \vee S^n \arrow[d, "f \perp g"] \\
 & X
\end{tikzcd}$$

to obtain $(f \perp g)_*(e_1) = f_*(\iota)$. Similary $(f \perp g)_*(e_2) = g_*(\iota)$. Therefore $h([fg]) = (fg)_*(\iota) = (f \perp g)_*(e_1 + e_2) = f_*(\iota) + g_*(\iota) = h[f] + h[g]$ and so h is a homomorphism.

We now specialize to the case $n = 1$.

Let exp be the generator of $S_1(S^1)$ defined by $\exp : \Delta^1 = I \to S^1$ where $\exp(t) = e^{2\pi i t}$. In S^1, set $v = 1 = \exp(0)$ and $w = -1 = \exp(1)$. Clearly $\partial(\exp) = v - v = 0$ so $[\exp]$ is a cycle and thus represents a homology class in $H_1(S^1)$.

Lemma 9.16.1 $[\exp]$ *is a generator of* $H_1(S^1)$.

Proof: Set $D := [-1, 1]$, $D^+ := [0, 1]$ and $D^- := [-1, 1]$. Let f be the composite $\Delta^1 = I \cong D^+ \xrightarrow{\tilde{f}} S^1$ where $\tilde{f}(t) = e^{\pi i t}$ and let g be the composite $\Delta^1 = I \cong D^- \xrightarrow{\tilde{g}} S^1$ where $\tilde{g}(t) = e^{\pi i (t+1)}$. We have isomorphisms

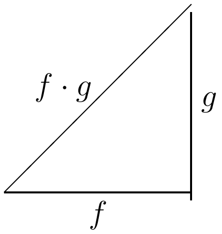
$$\begin{tikzcd}
H_1(D^+, S^0) \arrow[r, "\cong", "\text{excision}"'] \arrow[d, "\partial", "\cong"'] & H_1(S^1, D^-) \\
\mathbb{Z} \cong \tilde{H}_0(S^0) & H_1(S^1) \arrow[u, "\cong"']
\end{tikzcd}$$

$w - v$ is a generator of $\tilde{H}_0(S^0)$ so its image under the isomorphisms is a generator of $H_1(S^1)$. $f \in S_1(D^+)$ has the property that $\partial f = w - v$ so it represents the generator of $H_1(D^+, S^0)$

which hits $w - v$ under the isomorphism ∂, and thus its image in $S_1(S^1)/S_1(D^-)$ represents a generator of $H_1(S^1, D^-)$. $f + g \in S_1(S^1)$ projects to f in $S_1(S^1)/S_1(D^-)$, and $f + g$ is a cycle so the homology class $[f + g]$ is a generator of $H_1(S^1)$. Since $\exp = f \cdot g$, we conclude the proof by the following Lemma which shows that $[\exp] = [f + g]$.

Lemma 9.16.2 *Let $f, g : I \to X$ such that $g(0) = f(1)$. Then as elements of $S_1(X)$, $f \cdot g$ is homologous to $f + g$.*

Proof: Define $T : \Delta^2 \to X$ by extending the map shown around the boundary:



This is possible since the map around the boundary is null homotopic.

$\partial T = f - f \cdot g + g$, so $f \cdot g$ is homologous to $f + g$. □

We will use $[\exp]$ for ι_1.

Theorem 9.16.3 *(Baby Hurewicz Theorem)*

Suppose X is connected. Then $h : \pi_1(X) \to H_1(X)$ is onto and its kernel is the commutator subgroup of $\pi_1(X)$. ie. $H_1(X) \cong \pi_1(X)/(\text{commutator subgroup}) = \text{abelianization of } \pi_1(X)$.

Proof: Let x_0 be the basepoint or X.
Show that h is onto:

Let $z = \sum n_i T_i$ represent a homology class in $H_1(X)$. Thus $0 = \partial z = \sum n_i \big(T_i(1) - T_i(0)\big)$.

Let γ_{i0} and γ_{i1} be paths joining x_0 to $T_i(0)$ and $T_i(1)$ respectively.

Let $S_i = \gamma_{i0} + T_i - \gamma_{i1} \in S_1(X)$ (also writing $\gamma_{i\epsilon}$ for the composite $I \to I/\sim = S^1 \xrightarrow{\gamma_{i\epsilon}} X$). Thus $z = \sum n_i S_i$ since the γ's cancel out, using $\partial z = 0$. (Each γ_i appears equally often with $\epsilon = 0$ as with $\epsilon = 1$.)

By the preceding Lemma, f_i is homologous to $\gamma_{i0} + T_i - \gamma_{i1} = S_i \in S_1(X)$. Therefore $h(\prod f_i^{n_i}) = (\prod f_i^{n_i})_*(\iota_1) = [\sum f_i^{n_i}] = [\sum n_i S_i] = [z]$.
Show $\ker h =$ commutator subgroup:

$H_1(X)$ is abelian so (commutator subgroup) $\subset \ker h$.

Conversely, suppose $f \in \ker h$. Then, regarded as a generator of $S_1(X)$, $f = \partial z$ for some $z \in S_2(X)$.

Write $f = \partial(\sum n_i T_i) = \sum n_i \partial T_i$. Let $\partial T_i = \alpha_{i0} - \alpha_{i1} + \alpha_{i2}$ and for $j = 0, 1, 2$ choose paths γ_{ij} joining x_0 to the endpoints of α_{ij} as shown, making sure to always choose the smae path γ_{ij} if a given poiont occurs as an endpoint more than once.

Set
$g_{i0} := \gamma_{i1}\alpha_{i0}\gamma_{i2}^{-1}$
$g_{i1} := \gamma_{i0}\alpha_{i1}\gamma_{i2}^{-1}$
$g_{i0} := \gamma_{i0}\alpha_{i2}\gamma_{i1}^{-1}$
Set $g_i = g_{i0}g_{i1}^{-1}g_{i2} = \gamma_{i1}\alpha_{i0}\alpha_{i1}^{-1}\alpha_{i2}\gamma_{i1}^{-1}$.

Since $\alpha_{i0}\alpha_{i1}^{-1}\alpha_{i2}$ can be extended to a map on the interior (namely T_i,) it is null homotopic, so $g_i \simeq *$. Therefore $\prod_i (g_i)^{n_i} = 1 \in \pi_1(X)$. But $f = \sum n_i \partial T_i = \sum n_i(\alpha_{i0} - \alpha_{i1}^{-1} + \alpha_{i2})$ in the free abelian group $S_1(X)$. This means that when terms are collected on the right, f remains with coefficient 1 and all other terms cancel. Thus modulo the commutator subgroup the product $\pi_i(g_i)^{n_i}$ can be reordered to give f with the $\gamma's$ cancelling out. Therefore, modulo commutators, $f = 1$ so that $f \in$ (commutator subgroup). $\square$