Chapter 7

Homotopy

7.1 Basic concepts of homotopy

Example:

$$\int_{\gamma_1} \frac{1}{z} dz = \int_{\gamma_2} \frac{1}{z} dz$$

but

$$\int_{\gamma_1} \frac{1}{z} dz \neq \int_{\gamma_3} \frac{1}{z} dz.$$

Why? The domain of $1/z$ is $\mathbb{C} \smallsetminus \{0\}$. We can deform γ_1 continuously into γ_2 without leaving $\mathbb{C} \smallsetminus \{0\}$.

Intuitively, two maps are homotopic if one can be continuously deformed to the other.

The value of $\int_\gamma \frac{1}{z} dz$ is an example of a situation where only the homotopy class is important.

Definition 7.1.1 *Let X and Y be topological spaces, and $A \subset X$, and $f, g : X \to Y$ with $f|_A = g|_A$. We say f* is homotopic to *g* relative to *A (written $f \simeq g \operatorname{rel} A$) if $\exists H : X \times I \to Y$ s.t. $H|_{X\times 0} = f$, $H|_{X \times 1} = g$, and $H(a,t) = f(a) = g(a)$ $\forall a \in A$. H is called a homotopy from f to g.*

In the example, $X = I$, $Y = \mathbb{C} \smallsetminus \{0\}$, $A = \{0\} \cup \{1\}$, $f(0) = g(0) = p$, $f(1) = g(1) = q$.

Notation: For $t \in I$, $H_t : X \to Y$ by $H_t(x) = H(x,t)$. In other words $H_0 = f$, $H_1 = g$.

$f \overset{H}{\simeq} g$ rel A or $H : f \simeq g$ rel A mean H is a homotopy from f to g. We write $f \simeq g$ if A is understood.

Example: $Y = \mathbb{R}^n$, $f, g : X \to \mathbb{R}^n$. $f|_A = g|_A$. Then $f \simeq g$ rel A.
Proof: Define $H(x,t) = tg(x) + (1-t)f(x)$

Proposition 7.1.2 *$A \subset X$. $j : A \to Y$. Then homotopy rel A is an equivalence relation on $\mathcal{S} = \{f : X \to Y | f|_A = j\}$.*

Proof: (i) reflexive: given $f \in \mathcal{S}$, define $H : f \simeq f$ by $H(x,t) = f(x)\ \forall t$.

(ii) Symmetric: Given $H : f \simeq g$ define $G : g \simeq f$ by $G(x,t) = H(x, 1-t)$.

(iii) Transitive: Given $F : f \simeq g$, $G : g \simeq h$ define $H : f \simeq h$ by

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } 0 \le t \le 1/2 \\ G(x,2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

□

Important special case: $A =$ pt x_0 of X.

Definition 7.1.3 *A* pointed space *consists of a pair $\{X, x_0\}$. $x_0 \in X$ is called the* basepoint. *A* map of pointed spaces *$f : (X, x_0) \to (Y, y_0)$ is a map of pairs, in other words $f : X \to Y$ s.t. $f(x_0) = y_0$.*

Note: Pointed spaces and basepoint-preserving maps form a category.

Notation: X, Y pointed spaces. $[X, Y] = \{$ homotopy equivalence classes of pointed maps $\}$.

Top (X, Y) is far too large to describe except in trivial cases (such as $X =$ pt). But $[X, Y]$ is often countable or finite so that a complete computation is often possible. For this case under certain hypotheses (discussed later) this set has a natural group structure.

Notation: $\pi_n(Y, y_0) \overset{\text{def}}{=} [S^n, Y]$ with basepoints $(1, 0, \ldots, 0)$ and y_0 respectively. In this special case $X = S^n$, this set has a natural group structure (described later). $\pi_n(Y, y_0)$ is called the *n-th homotopy group of Y* with respect to the basepoint y_0.

$\pi_1(Y, y_0)$ is callsed the *fundamental group of Y* with respect to the basepoint y_0.

7.1.1 Group Structure of $\pi_1(Y, y_0)$

Notation: $f, g : I \to Y$. Suppose $f(1) = g(0)$.

Define $f \cdot g : I \to Y$ by

$$f \cdot g(s) = \begin{cases} f(2s) & \text{if } 0 \le s \le 1/2 \\ g(2s-1) & \text{if } 1/2 \le s \le 1 \end{cases}$$

Lemma 7.1.4 *$f, g : I \to Y$ s.t. $f(1) = g(0)$. $A = \{0\} \cup \{1\} \subset I$. Then the homotopy class of $f \cdot g$ rel A depends only on the homotopy classes of f and g rel A. In other words $f \simeq f'$ and $g \simeq g' \Rightarrow f \cdot g \simeq f' \simeq g'$.*

$F : f \simeq f'$, $G : g \simeq g'$.

$H : I \times I \to Y$

$$H(s,t) = \begin{cases} F(2s,t) & \text{if } 0 \le s \le 1/2 \\ G(2s-1,t) & \text{if } 1/2 \le s \le 1. \end{cases}$$

$H : f \cdot g \simeq f' \cdot g'$. $\square$

Let $f, g \in \pi_1(Y, y_0)$. So $f, g : S^1 \to Y$.

Thought of as maps $I \to Y$ for which $f(0) = f(1) = g(0) = g(1) = y_0$.

Define $f \star g$ in $\pi_1(Y, y_0)$ to be $f \cdot g$.

Theorem 7.1.5 *$\pi_1(Y, y_0)$ becomes a group under $[f][g] := [fg]$.*

Proof: The preceding lemma show that this multiplication is well defined.

Associativity:

Follows from:

Lemma 7.1.6 *Let $f, g, h : I \to Y$ such that $f(1) = g(0)$ and $g(1) = h(0)$. Then $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$,*

Proof: Explicitly $H(s,t) = \begin{cases} f(\frac{4s}{2-t}) & 4s \le 2-t; \\ g(4s+t-2) & 2-t \le 4s \le 3-t; \\ h(\frac{4s+t-3}{1+t}) & 3-t \le 4s. \end{cases}$ $\surd$

Identity: Given $y \in Y$, define $c_y : I \to Y$ by $c(s) = y$ for all s. Constant map.

Lemma 7.1.7 *Let $f : I \to Y$ be such that $f(0) = p$. Then $c_p \cdot f \simeq f \operatorname{rel}(\{0\} \cup \{1\})$.*

$$H(s,t) = \begin{cases} p & 2s \le t; \\ f(\frac{2s-t}{2-t}) & 2s \ge t. \end{cases}$$ $\square$

Similarly if $f(1) = q$ then $f \cdot c_q \simeq f \operatorname{rel} A$. Applying this to the case $p = q = y_0$ gives that $[f][c_{y_0}] = [c_{y_0}][f] = [f]$. $\surd$

Inverse: Let $f : I \to Y$ Define $f^{-1} : I \to Y$ by $f^{-1}(s) := f(1-s)$.

Lemma 7.1.8 *. $f \cdot f^{-1} \simeq c_p \operatorname{rel}(\{0\} \cup \{1\})$.*

Proof: Intuitively:

$t = 1$ Go from p to q and return.

$0 < t < 1$ Go from p to $f(t)$ and then return.

$t = 0$ Stay put.

$$H(s,t) = \begin{cases} f(2st) & 0 \le s \le 1/2; \\ f\big(2(1-s)t\big) & 1/2 \le s \le 1. \end{cases}$$ $\square$

Applying the lemma to the case $p = q = y_0$ shows $[f][f^{-1}] = [c_{y_0}]$ in $\pi_1(Y, y_0)$, $\surd$

This completes the proof that $\pi_1(Y, y_0)$ is a group under this multiplication. $\square$

Note: In general $\pi_1(Y, y_0)$ is nonabelian.

Proposition 7.1.9 *Let* $f : X \to Y$ *be a pointed map. Define* $f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ *by* $f_\#[\omega] := [f \circ \omega]$. *Then* $f_\#$ *is a group homomorphism.*

($f_\#$ is called the map *induced* by f.)

Proof:
Show that $f_\#$ is well defined.

Lemma 7.1.10

$$\begin{tikzcd}(W,A) \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & (X,B) \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & (Y,C)\end{tikzcd}$$

Suppose $g \simeq g' \operatorname{rel} A$ *and* $h \simeq h' \operatorname{rel} B$. *Then* $h \circ g \simeq h' \circ g' \operatorname{rel} A$.

Proof of Lemma:
Let $G : g \simeq g'$ and $H : h \simeq h'$ be the homotopies. Define $K : W \times I \to Y$ by $K(w,t) := H\big(G(w,t), t\big)$. Then $K : h \circ g \simeq h' \circ g' \operatorname{rel} A$. (i.e. $K(w,0) = H\big(G(w,0),0\big) = H\big(g(w),0\big) = h \circ g(w)$ and similarly $K(w,1) = h' \circ g'(w)$ while for $a \in A$, $K(a,t) = H\big(G(a,t),t\big) = H\big(g(a),t\big) = h\big(g(a)\big) = h'\big(g'(a)\big)$.

Proof of Proposition (cont.) Thus $f_\#$ is well defined (applying the lemma with $W := S^1$, $A = \{w_0 := (1,0)\}$, $B := \{x_0\}$, $C := \{y_0\}$, $g := w$, $g' := w'$, and $h = h' := f$). $\surd$

$f \circ (w \cdot \gamma) = (f \circ \omega) \cdot (f \circ \gamma)$ Therefore $f_\#([\omega][\gamma]) = f_\#([\omega \cdot \gamma]) = [f \circ (\omega \cdot \gamma)] = [(f \circ \omega) \cdot (f \circ \gamma)] = [f \circ \omega][f \circ \gamma] = f_\#([\omega]) f_\#([\gamma])$. $\square$

Corollary 7.1.11 *The associations* $(X, x_0) \longmapsto \pi_1(X, x_0)$ *with* $f \longmapsto f_\#$ *defines a functor from the category of pointed topological spaces to the category of groups.* $\square$

To what extent does $\pi_1(Y, y_0)$ depend on y_0?

Proposition 7.1.12

1. *Let* Y' *be the path component of* Y *containing* y_0. *Then* $\pi_1(Y', y_0) \simeq \pi_1(Y, y_0)$.
2. *If* y_0 *and* y_1 *are in the same path component then* $\pi_1(Y, y_0) \simeq \pi_1(Y, y_1)$

Proof:

1. Any curve of Y beginning at y_0 lies entirely in Y' (since curves are images of a path connected set and thus path connected).

2. Pick a path α joining y_0 to y_1. Define $\phi : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ by $[f] \mapsto [\alpha^{-1} \cdot f \cdot \alpha]$ (where α^{-1} denotes the path which goes backwards along α).

   Check that ϕ is a homorphism:

   $\phi([f][g]) = [\alpha^{-1} f \alpha][\alpha^{-1} g \alpha] = [\alpha^{-1} f \alpha \alpha^{-1} g \alpha] = [\alpha^{-1} f g \alpha]$ since $f \alpha \alpha^{-1} g \simeq f c_{y_1} g \simeq fg$. Thus $\phi([f][g]) = [\alpha^{-1} f g \alpha] = \phi([fg])$ $\checkmark$

   Show ϕ is injective:

   Suppose that $\phi([f]) = e$. That is $[\alpha^{-1} f \alpha] = [c_{y_1}]$. Then $\alpha^{-1} f \alpha \simeq c_{y_1}$. Hence $f \simeq c_{y_0} f c_{y_0} \simeq \alpha \alpha^{-1} f \alpha \alpha^{-1} \simeq \alpha c_{y_1} \alpha^{-1} \simeq \alpha \alpha^{-1} \simeq c_{y_0}$. Thus $[f] = [e]$ in $\pi_1(Y, y_0)$. $\checkmark$

   Check that ϕ is onto:

   Given $[g] \in \pi_1(Y, y_1)$, set $f := \alpha \cdot g \cdot \alpha^{-1}$. Then $\phi[f] = [\alpha^{-1} f \alpha] = [\alpha^{-1} \alpha g \alpha^{-1} \alpha] = [g]$. $\checkmark$

□

In algebraic topology, path connected is a more important concept than connected. From now on, we will use the term “connected” to mean “path connected” unless stated otherwise.

Notation: If Y is (path) connected, write $\pi_1(Y)$ for $\pi_1(Y, y_0)$ since up to isomorphism it is independent of y_0. The constant function $(X, x_0) \to (Y, y_0)$ taking x to y_0 for all $x \in X$ is often denoted $*$. Also the basepoint itself is often denoted $*$.

If $f \simeq *$ then f is called *null homotopic*. So for $f : S^1 \to Y$, f is null homotopic if and only if $[f] = e$ in $\pi_1(Y)$.

Theorem 7.1.13 *Let* $X = \prod_{j \in I} X_j$. *Let* $* = (x_j)_{j \in I} \in X$. *Then* $\pi_1(X, *) = \prod_{j \in I} \pi_1(X_j, x_j)$.

Proof:

Let $p_j : X \to X_j$ be the projection. The homomorphisms $p_{j\#} : \pi_1(X, *) \to \pi_1(X_j, x_j)$ induce $\phi := (p_{j\#}) : \pi_1(X, *) \to \prod_{j \in I} \pi_1(X_j, x_j)$.

To show ϕ injective:

Suppose that $\phi([\omega]) = 1$. Then $\forall j \in I$, $\exists$ a homotopy $H_j : p_j \circ \omega \simeq c_{x_j}$. Put these together to get $H : \omega \simeq c_*$. (i.e. for $z = (z_j)_{j \in I} \in X$, define $H(z, t) := \big(H_j(z_j, t)\big)_{j \in I}$ Hence $[\omega] = 1$ in $\pi_1(X, *)$.

To show ϕ surjective:

Given $([\omega_j])_{j \in I}$ where $[\omega_j] \in \pi_1(X_j, x_j)$:

Define ω to be the path whose jth component is ω_j. (That is, $\omega(t) = \big(\omega_j(t)\big)_{j \in I}$.) Then $\phi([\omega]) = ([\omega_j])_{j \in I}$. □

Definition 7.1.14 *If X is (path) connected and $\pi_1(X) = 1$ (where 1 denotes the group with just one element) then X is called* simply connected.