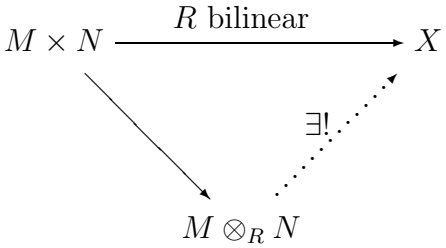
9.11 Homology and Cohomology with Coefficients

9.11.1 Tensor Product

Let R be a commutative ring and let M and N be R-modules.

The *tensor product* $M \otimes_R N$ is the R-module with the universal property



Explicity, $M \otimes_R N = F_{ab}(M \times N)/\sim$ where

$(m, n_1 + n_2) \sim (m, n_1) + (m, n_2)$

$(m_1 + m_2, n) \sim (m_1, n) + (m_2, n)$

$(mr, n) \sim (m, rn)$

with the R-modules structure $f(m, n) := (rm, n) = (m, rn)$,

$[(m, n)]$ in $M \otimes_R N$ is written $m \otimes n$.

Thus elements of $M \otimes_R N$ are of the form $\sum_{i=1^k} m_i \otimes n_i$.

9.11.2 Coefficients

Let C be a chain complex and let G be an abelian group. Define a chain complex denoted $C \otimes G$ by $C \times G)_p := C_p \otimes G$ with boundary operator defined to be $d \times 1_G : C_p \otimes G \to C_{p-1} \otimes G$, where d is the boundary operator on C. Similarly if C is a cochain complex, can define a cochain complex $C \otimes G$ by $C \otimes G)^p := C^p \otimes G$ with boundary operator $d \otimes 1_G$.

There is a version of the Universal Coefficient Theorem which gives the homology (resp. cohomology) of $C \otimes G$ in terms of the homology (resp. cohomology) of C whenever C is either free abelian or G is free abelian. However we will now give a direct proof that if C, D are free chain complexes and $\phi : C \to D$ s.t. $\phi_* : H_*(C) \to H_*(D)$ is an isomorphism then $\phi_* \otimes G : H_*(C \otimes G) \to H_*(D \otimes G)$ is an isomorphism.

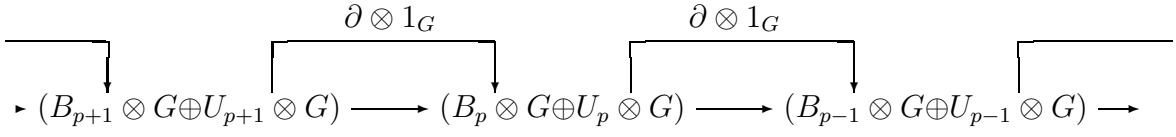
Proposition 9.11.1 *Let C be a free chain complex s.t. $H_q(C) = 0 \; \forall q$. Then $H_q(C \otimes G) = 0 \; \forall q$.*

Proof: As in the proof that $H^q(\operatorname{Hom} C, \mathbb{Z}) = 0$, we can describe C as follows:

$$C \qquad \xrightarrow{\partial} (B_{p+1} \oplus U_{p+1}) \xrightarrow{\partial} (B_p \oplus U_p) \xrightarrow{\partial} (B_{p-1} \oplus U_{p-1}) \xrightarrow{\partial}$$

where $C_p \cong B_p \oplus U_p$ with $\partial_p : U_p \cong B_{p-1}$.
Therefore

$C \otimes G$



so $H_p(C) = 0 \ \forall p$. □

Proposition 9.11.2 *Let* $0 \to C \xrightarrow{\phi} D \to E \to)$ *be a short exact sequence of chain complexes s.t. E is a free chain complex. If* $\phi_* : H_q(C) \xrightarrow{\cong} H_q(D) \ \forall q$ *then* $\phi_* \otimes G : H_q(C \otimes G) \to H_q(D \otimes G)$ *is an isomorphism* $\forall q$.

Proof: Since E_p is free $\forall\ p$, $D_p \cong C_p \oplus E_p$ and thus $D_p \otimes G \cong C_p \otimes G \oplus E_p \otimes G$.

Hence $0 \to C \otimes G \xrightarrow{\phi \otimes G} D \otimes G \longrightarrow E \otimes G \to 0$ is again a short exact sequence so $H_q(E) = 0 \ \forall q \Rightarrow H_q(E \otimes G) = 0 \forall q \Rightarrow \phi_q \otimes G$ is an isomorphism $\forall q$. □

Without the freeness condition, $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/(2\mathbb{Z}) \to 0$ is exact but tensoring with $G = \mathbb{Z}/(2\mathbb{Z})$ gives $0 \to \mathbb{Z}/(2/\mathbb{Z}) \xrightarrow{2} \mathbb{Z}/(2/ZZ) \to \mathbb{Z}/(2\mathbb{Z}) \to 0$ which is not exact.

Theorem 9.11.3 *Let C, D be free chain complexes such that* ϕ_* *is an isomorphism on (co)homology* $\forall q$. *Then* $\phi_* \otimes G$ *is an isomorphism on (co)homology* $\forall\ q$.

Proof: The homology case follows from the preceding propositions, given the earlier theorem on existence of algebraic mapping cones. This also proves the cohomology statement, since a cochain complex is merely a chain complex with the groups renumbered.

For a simplicial complex K, we define the simplicial homology of K with coefficients in G, denoted $H_*(K;G)$ by $H_*(K;G) := H_*\big(C_*(K) \otimes G\big)$. Similarly if X is a topological space, its singular homology with coefficients in G is defined by $H_*(X;G) := H_*\big(S_*(X) \otimes G\big)$ and if X is a CW-comples, its cellular homology with coefficients in G is $H_*\big(D_*(X) \otimes G\big)$. Can likewise define $H^*(K;G) := H_*\big(C^*(K) \otimes G\big)$. $H^*(X;G) := H_*\big(S^*(X) \otimes G\big)$ and cellular cohomology of a CW-complex X as $H^*\big(S^*(X) \otimes G\big)$. We can also define relative and reduced homology and cohomology groups with coefficients in G.

From the preceding theorem we get $H_*(K;G) := H_*\big(|K|;G\big)$ and $H^*(K;G) := H^*\big(|K|;G\big)$ and $H_*(D(X);G) := H_*(X;G$ and $H^*(D(X);G) := H^*(X;G$. It is also immediate that $H_*(X;G)$ and $H^*(X;G)$ satisfy all the axioms for a homology (resp. cohomology) theory except for $A7$ which has to be replaced by

$$H_p(*;G) = \begin{cases} 0 & p \neq 0; \\ G & p = 0; \end{cases} \qquad H^p(*;G) = \begin{cases} 0; & p \neq 0 \\ G & p = 0. \end{cases}$$

Similarly Mayer-Vietoris works, Also $\tilde{H}_*(X;G)$ satisfies

$$H_n(X;G) = \begin{cases} \tilde{H}(X;G) & n > 0; \\ H_0(G) \oplus G & n = 0; \end{cases}$$

and $\tilde{H}_n(X;G) \cong H_n\big((X,*);G\big)$. The cohomology versions work also.

If $G \to H$ is a homomorphism of abelian groups, then it induces a (co)chain map $C \otimes G \to C \otimes H$ for any (co)chain complex C and thus induces $H_*(X;G) \to H_*(X;H)$ and $H^*(X;G) \to H^*(X;H)$ (notice that the direction of latter arrow does not get reversed).

If G happens to have an R-module structure for some commutative ring R (with 1) then for any abelian group A, $A \otimes G$ becomes an R-module by defining on generators $r(a \otimes g) := a \otimes rg$. In this case, for $c \in C_p$¡ $g \in G$:

$r\big(\partial(c \otimes g)\big) = r(\partial c \otimes g) = \partial c \otimes rg = \partial(c \otimes rg) = \partial\big(r(c \otimes g)\big)$. That is, the boundary operator on $C \otimes G$ becomes an R-module homomorphism, so $\ker \partial$ and $\operatorname{Im} \partial$ are R-modules and so their quotient, $H_*(C \otimes G)$ inherits an R-module structure.

Suppose now that G is a ring R (commutative, with 1) and C is a free chain complex. The Kronecker product induces a bilinear pairing between $C \otimes R$ and $\operatorname{Hom}(C,\mathbb{Z}) \otimes R$ with values in R, which is again called the Kronecker product. Explicitly, given generators $f \otimes r$ of $\operatorname{Hom}(C,\mathbb{Z}) \otimes R$ and $c \otimes r'$ of $C \otimes R$, $\langle f \otimes r, c \otimes r' \rangle := rr\langle f, c\rangle$, where the multiplication takes place in R after taking the image of the integer-valued Kronecker prduct $\langle f, c\rangle$ under the unique ring homomorphism $\mathbb{Z} \to R$ (sending $1 \in \mathbb{Z}$ to $1 \in R$). This results in a bilinear R-module pairing (also called the Kronecker product) between the homology and cohomology groups as well.

We can also define cup products on cohomology with coefficients in R. Namely, for generators $f \otimes f \in S^p(X;R)$ and $g \otimes f' \in S^q(X;R)$ define $(f \otimes f) \cup (g \otimes r') \in S^{p+q}(X;R)$ by $(f \otimes f) \cup (g \otimes r') := (f \cup g) \otimes rr$;. Thus $S^*(X;R)$ and $H^*(X;R)$ become graded rings (with 1) and $H^*(X;R)$ is graded commutative. If $A \to R$ is a ring homomorphism then it follows immediately from the definitions that $S^*(X;A) \to S^*(X;R)$ and $H^*(X;A) \to H^*(X;R)$ are ring homomorphisms. (Note the special case were $A = \mathbb{Z} \to R$ given by $1 \mapsto 1$).

Given generators $\otimes r \in S^q(X;R)$ and $x \otimes r' \in S_{p+q}(X)$, can define cap product by $(g \otimes r) \cap (x \otimes r') := (g \cap x) \otimes rr$. Similarly one can define the relative cup and cap products.

Remark 9.11.4 *In practice, there are sometimes advantages to having a field as coefficients. Thus, besides $\mathbb{Z}$, the most common coefficients are $\mathbb{Z}/(p\mathbb{Z})$ and $\mathbb{Q}$. Sometimes $R = \mathbb{Z}_{(p)}$, $\mathbb{R}$, or $\mathbb{C}$ are also useful.*

Theorem 9.11.5

$$H_q(S^n;R)=\begin{cases}R & q=0,n;\\ 0 & q\neq 0,n;\end{cases}\qquad H^q(S^n;R)=\begin{cases}R & q=0,n;\\ 0 & q\neq 0,n;\end{cases}$$

$$H_q(\mathbb{C}P^n;R)=\begin{cases}R & q \text{ even, } q\leq 2n;\\ 0 & q \text{ odd or } q>2n;\end{cases}\qquad H^q(\mathbb{C}P^n;R)=\begin{cases}R & q \text{ even, } q\leq 2n;\\ 0 & q \text{ odd or } q>2n;\end{cases}$$

Proof: Use cellular (co)homology. e.g.
$D_*(\mathbb{C}P^n\otimes R)$ $\qquad R\to 0\to R\to 0\to R\to\ldots\to R\to 0\to R\to 0$ □

Theorem 9.11.6

$$H_q(\mathbb{R}P^n;\mathbb{Z}/(2\mathbb{Z}))=\begin{cases}\mathbb{Z}/(2\mathbb{Z}) & q\leq n;\\ 0 & q>n;\end{cases}$$
$$H^q(\mathbb{R}P^n;\mathbb{Z}/(2\mathbb{Z}))=\begin{cases}\mathbb{Z}/(2\mathbb{Z}) & q\leq n;\\ 0 & q>n;\end{cases}$$
$$H_q(\mathbb{R}P^n;\mathbb{Q})=H^q(\mathbb{R}P^n;\mathbb{Q})=\begin{cases}\mathbb{Q} & q=n \text{ when } n \text{ is even, or } q=0;\\ 0 & \text{otherwise.}\end{cases}$$

Proof: Use cellular (co)homology.
$D_*(\mathbb{R}P^n)$ $\qquad \mathbb{Z}\to\mathbb{Z}\to\ldots\xrightarrow{0}\mathbb{Z}\xrightarrow{2}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\to 0$
Therefore
$D_*(\mathbb{R}P^n\otimes\mathbb{Z}/(2\mathbb{Z}))$
$$\mathbb{Z}/(2\mathbb{Z})\to\mathbb{Z}/(2\mathbb{Z})\to\ldots\xrightarrow{0}\mathbb{Z}/(2\mathbb{Z})\xrightarrow{2=0}\mathbb{Z}/(2\mathbb{Z})\xrightarrow{0}\mathbb{Z}/(2\mathbb{Z})\to 0$$
Thus $H_q(\mathbb{R}P^n;\mathbb{Z}/(2\mathbb{Z}))=\begin{cases}\mathbb{Z}/(2\mathbb{Z}) & q\leq n;\\ 0 & q>n,\end{cases}$

$D_*(\mathbb{R}P^n\otimes\mathbb{Q})$ $\qquad \mathbb{Q}\to\mathbb{Q}\to\ldots\xrightarrow{0}\mathbb{Q}\xrightarrow[\cong]{2}\mathbb{Q}\xrightarrow{0}\mathbb{Q}\to 0$

Since $2:\mathbb{Q}\to\mathbb{Q}$ is an isomorphism (with mult. by $1/2$ as inverse), $H_*(\mathbb{R}P^n;\mathbb{Q})$ is as stated. Similarly one gets the cohomology results.

Remark 9.11.7 *If R is a field, then it follows from the Universal Coefficient Theorem that* $H^*(X;R)\cong \mathrm{Hom}_{R-\mathrm{mods}}\big(H_*(X,R),R\big)$.