Chapter 9

Homological Algebra

9.1 Introductory concepts of homological algebra

Definition 9.1.1 *A* chain complex (C, d) *of abelian groups consists of an abelian group* C_p *for each integer* p *together with a morphism* $d_p : C_p \to C_{p-1}$ *for each* p *such that* $d_{p-1} \circ d_p = 0$*. Maps* d_p *are called* boundary operators *or* differentials*.*

The subgroup $\ker d_p$ *of* C_p *is denoted* $Z_p(C)$*. Its elements are called* cycles*.*

The subgroup $\operatorname{Im} d_{p+1}$ *of* C_p *is denoted* $B_p(C)$*. Its elements are called* boundaries*.*

$d_p \circ d_{p+1} = 0 \Rightarrow B_p(C) \subset Z_p(C)$.

The quotient group $Z_p(C)/B_p(C)$ is denoted $H_p(C)$ and called the p-th *homology group* of C. Its elements are called *homology classes.*

$x, y \in C_p$ are called *homologous* if $x - y \in B_p(C)$.

Definition 9.1.2 *A* chain map $f : C \to D$ *consists of a group homomorphism* f_p $\forall p$ *s.t.*

$$\begin{tikzcd}
C_p \arrow[r, "d_p"] \arrow[d, "f_p"] & C_{p-1} \arrow[d, "f_{p-1}"] \\
D_p \arrow[r, "d_p"] & D_{p-1}
\end{tikzcd}$$

Notation: The subscripts are often omitted, so we might write $d^2 = 0$ or $fd = df$.

Remark: The composition of chain maps is a chain map so chain complexes and chain maps form a category.

A chain map $f : C \to D$ induces a homomorphism $f_* : H_p(C) \to H_p(D)$ for all p, defined as follows:

Let $x \in Z_p(C)$ represent an element $[x] \in H_p(C)$.

Then $df(x) = fd(x) = f(0) = 0$ so $f(x) \in Z_p(D)$.

Define $f_*([x]) := [f(x)]$.

If x, x' represent the same element of $H_p(C)$ then $x - x' = dy$ for some $y \in C_{p+1}(C)$. Therefore $fx - fx' = fdy = d(fy)$ which implies $f(x), f(x')$ represent the same element of $H_p(D)$. So f_* is well defined.

Definition 9.1.3 *A composition of homomorphisms of abelian groups*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

is called exact *at* Y *if* $\ker g = \operatorname{Im} f$. *A sequence*

$$\begin{tikzcd} X_n \arrow[r, "f_n"] & X_{n-1} \arrow[r, "f_{n-1}"] & \dots \arrow[r] & X_1 \arrow[r, "f_1"] & X_0 \end{tikzcd}$$

is called exact *if it is exact at* X_i *for all* $i = 1, \dots, n-1$.

Remark: An exact sequence can be thought of as a chain complex whose homology is zero. More generally, homology can be thought of as the deviation from exactness.

A chain complex whose homology is zero is called *acyclic.*

Definition 9.1.4 *A 5-term exact sequence of the form*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

is called a short exact sequence.

Proposition 9.1.5 *Let*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence. Then f *is injective,* g *is surjective and* $B/A \cong C$.

Proof:

Exactness at $A \Rightarrow$ Ker $f =$ Im $(0 \to A) = 0 \Rightarrow f$ injective

Exactness at $C \Rightarrow$ Im $g =$ Ker $(C \to 0) = C \Rightarrow g$ surjective

Exactness at $B \Rightarrow B/\ker g \cong \operatorname{Im} g = C \Rightarrow B/\operatorname{Im} f \cong B/A$.

Corollary 9.1.6

(a) $0 \to A \xrightarrow{f} B \to 0$ *exact* $\Rightarrow f$ *is an isomorphism.*

(b) $0 \to A \to 0$ *exact* $\Rightarrow A = 0$.

Definition 9.1.7

A map $i : A \to B$ *is called a* split monomorphism *if* $\exists s : B \to A$ *s.t.* $si = 1_A$.

A map $p : A \to B$ *is called a* split epimorphism *if* $\exists s : B \to A$ *s.t.* $ps = 1_B$.

Note: The splitting s (should it exist) is not unique.

It is trivial to check:

(1) A split monomorphism is a monomorphism

(2) A split epimorphism is an epimorphism

Proposition 9.1.8 *The following are three conditions (1a, 1b, and 2) are equivalent:*

1. $\exists$ *a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *s.t.*

   1a) i is a split monomorphism

   1b) p is a split epimorphism

2. $B \cong A \oplus C$.

Remark: The isomorphism in 2. will depend upon the choice of splitting s in 1a (respectively 1b).

Lemma 9.1.9 (Snake Lemma) *Let*

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i'"] \arrow[d, "f'"] & A \arrow[r, "i''"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "j'"] & B \arrow[r, "j''"] & B'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram in which the rows are exact. Then $\exists$ *a long exact sequence*

$$0 \to \ker f' \to \ker f \to \ker f'' \xrightarrow{\partial} \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f'' \to 0.$$

Proof:

Step 1. Construction of the map ∂ (called the "connecting homomorphism"):

Let $x \in \ker f''$. Choose $y \in A$ s.t. $i''(y) = x$. Since $j''fy = f''i''y = f''x = 0$, $fy \in \ker j'' = \mathrm{Im} j'$ so $fy = j'(z)$ for some $z \in B'$. Define $\partial x = [z]$ in coker f'.

Show ∂ well defined:

Suppose $y, y' \in A$ s.t. $i''y = x = i''y'$.

$i''(y - y') = 0 \Rightarrow y - y' = i'(w)$ for some $w \in A'$. Hence $fy - fy' = fi'w = j'f'w$.

Therefore if we let $fy = j'z$ and $fy' = j'z'$ then $j'(z - z') = j'f'w \Rightarrow z - z' = f'w$ (since j is an injection). So $[z] = [z']$ in Coker f'. $\surd$

Step 2: Exactness at Ker f'':

Show the composition $\ker f \xrightarrow{i''} \ker f'' \xrightarrow{\partial}$ Coker f' is trivial.

Let $k \in$ Ker f. Then $\partial(i''k) = [z]$ where $j'(z) = f(k) = 0$. So $z = 0$.

So $\partial \circ i'' = 0$. Hence Im $(i'') \subset$ Ker ∂.

Conversely let $x \in$ Ker ∂. Let $y \in A$ s.t. $i''y = x$. We wish to show that we can replace y by a $y' \in \ker f$ which satisfies $i''y' = x$.

Find $z \in B'$ s.t. $j'z = fy$. So $\partial x = [z]$. $\partial x = 0 \Rightarrow z \in$ Coker f'.

Hence $z = f'w$ for some $w \in A'$.

Set $y' := y - i'w$. Then $i''y = iy - i''i'w = iy = x$ and $fy' = fy - fi'w = fy - j'f'w = fy - j'z = 0$.

Hence $y' \in$ Ker f.

The rest of the proof is left as an exercise □

Lemma 9.1.10 (5-Lemma)

Let

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "i"] & E \arrow[d, "j"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows. If f, g, i, j are isomorphisms then h is also an isomorphism.

(Actually , we need only f mono and j epi with g and i iso.)

Definition 9.1.11 *A sequence*

$$0 \to \underline{C} \xrightarrow{f} \underline{D} \xrightarrow{g} \underline{E} \to 0$$

of chain complexes and chain maps is called a short exact sequence of chain complexes *if*

$$0 \to C_p \xrightarrow{f_p} D_p \xrightarrow{g_p} E_p \to 0$$

is a short exact sequence (of abelian groups) for each p*.*

Theorem 9.1.12 *Let*

$$0 \to \underline{P} \xrightarrow{f} \underline{Q} \xrightarrow{g} \underline{R} \to 0$$

be a short exact sequence of chain complexes. Then there is an induced natural (long) exact sequence

$$\cdots \to H_n(P) \xrightarrow{f_*} H_n(Q) \xrightarrow{g_*} H_n(R) \xrightarrow{\partial} H_{n-1}(P) \xrightarrow{f_*} H_{n-1}(Q) \to \dots$$

Remark 9.1.13 *Natural means:*

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P' \arrow[r] & Q' \arrow[r] & R' \arrow[r] & 0
\end{tikzcd}$$

implies

$$\begin{tikzcd}
\dots \arrow[r] & H_n(P) \arrow[r] \arrow[d] & H_n(Q) \arrow[r] \arrow[d] & H_n(R) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(P) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_n(P') \arrow[r] & H_n(Q') \arrow[r] & H_n(R') \arrow[r, "\partial"] & H_{n-1}(P') \arrow[r] & \dots
\end{tikzcd}$$

Proof:

1. Definition of ∂*:*

Let $[r] \in H_n(R)$, $r \in Z_n(R)$*. Find* $q \in Q_n$ *s.t.* $g(q) = r$*.*

$g(dq) = d(qd) = dr = 0$ *(since* $r \in Z_n(R)$*), which implies* $dg = fp$ *for some* $p \in P_{n-1}$*.*

$f(dp) = dfp = d^2q = 0 \Rightarrow dp = 0$ *(as* f *injective).*

So $p \in Z_{n-1}(p)$*. Define* $\partial[r] = [p]$*.*

2. ∂ *is well defined:*

(a) Result is independent of choice of q*:*

Suppose $g(q) = g(q') = r$.
$g(q-q') = 0 \Rightarrow q - q' = f(p'')$ *for some* $p'' \in P_n$.
Find p' *s.t.* $dq' = fp'$.
$f(p-p') = d(q-q') = dfp'' = fdp'' \Rightarrow p - p' = dp'' \in B_{n-1}(P)$.
So $[p] = [p']$ *in* $H_{n-1}(P)$.

(b) Result is independent of the choice of representative for $[r]$*:*
Suppose $r' \in Z_n(R)$ *s.t.* $[r'] = [r]$.
$r - r' = dr''$ *for some* $r'' \in R_{n+1}$.
Find $q'' \in Q_{n+1}$ *s.t.* $gq'' = r''$.
$gdq'' = dgq'' = dr'' = r - r' = g(q) - r' \Rightarrow r' = g(q - dq'')$.
Set $q' := q - dq'' \in Q_n$.
$gq' = r'$ *so we can use* q' *to compute* $\partial[r']$.
$dq' = dq - d^2q'' = dq$ *so the definition of* $\partial[r']$ *agrees with the definition of* $\partial[r]$.

3. *Sequence is exact at* $H_{n-1}(P)$.

To show that the composition $H_n(R) \xrightarrow{\partial} H_{n-1}(P) \xrightarrow{f_*} H_{n-1}(Q)$ *is trivial:*
Let $[r] \in H_n(R)$. *Find* $q \in Q_n$ *s.t.* $gq = r$.
Then $\partial[r] = [p]$ *where* $fp = dq$.
So $f_*\partial[r] = [fp] = [dq] = 0$ *since* $dq \in B_{n-1}(Q)$.
Hence Im $\partial \subset$ *Ker* f_*.
Conversely let $[p] \in$ *Ker* f_*.
Since $[fp] = 0$, $fp = dq$ *for some* $q \in Q_n$.
Let $r = gq$. *Then* $\partial[r] = [p]$.
So Ker $f_* \subset$ *Im* ∂.

The proof of exactness at the other places is left as an exercise. □

Definition 9.1.14 *Let* $f, g : C \to D$ *be chain maps.*

A collection of maps $s_p : C_p \to D_{p+1}$ *is called a* chain homotopy *from* f *to* g *if the relation* $ds + sd = f - g : C_p \to D_p$ *is satisfied for each* p. *If there exists a chain homotopy from* f *to* g, *then* f *and* g *are called* chain homotopic.

Proposition 9.1.15 *Chain homotopy is an equivalence relation.*

Proof: *Exercise* □

Proposition 9.1.16 $f \simeq f'$, $g \simeq g' \Rightarrow gf \simeq g'f'$.

Proof: $\begin{tikzcd} \underline{C} \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & \underline{D} \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & \underline{E} \end{tikzcd}$

Show $gf \simeq gf'$*:*
Let $s : f \simeq f'$. $s : C_p \to D_{p+1}$ *s.t.* $ds + sd = f' - f$.
$g \circ s : C_p \to E_{p+1}$ *satisfies* $dgs + gsd = gds + gsd = g(ds + sd) = g(f' - g) = gf' - gf$.
Similarly $g'f \simeq g'f'$. ☐

Definition 9.1.17 *A map* $f : C \to D$ *is a* chain (homotopy) equivalence *if* $\exists g : D \to C$ *s.t.* $gf \simeq 1_C$, $fg \simeq 1_D$.

Proposition 9.1.18 $f \simeq g \Rightarrow f_* = g_* : H_*(C) \to H_*(D)$.

Proof: *Let* $[x] \in H_p(C)$ *be represented by* $x \in Z_p(C)$. *Let* $s : f \simeq g$.
Then $fx - gx = sdx + dsx = dsx \in B_p(C)$. *So* $[fx] = [gx] \in H_p(D)$. ☐

Corollary 9.1.19 $f : C \to D$ *is a chain equivalence* $\Rightarrow f_* : H_*(C) \to H_*(D)$ *is an isomorphism.* ☐

Proposition 9.1.20 **(Algebraic Mayer-Vietoris)** *Let*

$$\begin{tikzcd}
{} \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\alpha"] & B_n \arrow[r, "j"] \arrow[d, "\beta"] & C_n \arrow[r, "\partial"] \arrow[d, "\gamma"] & A_{n-1} \arrow[r, "i"] \arrow[d, "\alpha"] & B_{n-1} \arrow[r, "j"] \arrow[d, "\beta"] & C_{n-1} \arrow[r] \arrow[d, "\gamma"] & {} \\
{} \arrow[r] & A'_n \arrow[r, "i'"] & B'_n \arrow[r, "j'"] & C'_n \arrow[r, "\partial"] & A'_{n-1} \arrow[r, "i'"] & B'_{n-1} \arrow[r, "j'"] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

be a commutative diagram with exact rows. Suppose $\gamma : C_n \to C'_n$ *is an isomorphism* $\forall n$. *Then there is an induced long exact sequence*

$$\begin{tikzcd}
\ldots \arrow[r] & A_n \arrow[r, "\rho"] & B_n \oplus A'_n \arrow[r, "q"] & B'_n \arrow[r, "\Delta"] & A_{n-1} \arrow[r] & B_{n-1} \oplus A'_{n-1} \arrow[r] & B'_{n-1}
\end{tikzcd}$$

where
$\rho(a) = (ia, \alpha a)$
$q(b, a') = \beta b - i'a'$
$\Delta = \partial \gamma^{-1} j'$

Proof: *Exercise* ☐