9.10 Cup Products

From the last section (and the Universal Coefficient Theorem), we know that $H^*(X)$ is completely determined by $H_*(X)$, so why bother with cohomology at all? In any potential applicaiton, why not just use homology instead? One answer is that there is a natural way to put a multiplication called the "cup product" on $H^*(X)$ so that $H^*(X)$ becomes a ring. This might be used, for example, in a case where the $H^*(X)$ and $H^*(Y)$ to show that $X \not\approx Y$ if it should turn out that the multiplications on $H^*(X$ and $H^*(Y)$ were different.

Let $f \in S^p(X)$ and $S^q(X)$. Define $f \cup g \in S^{p+1}(X)$ as follows.
For a generator $T : \Delta^{p+q} \to X$ of $S_{p+q}(X)$ we define
$\langle f \cup g, T\rangle := (-1)^{pq}\big\langle f, T \circ l(\epsilon_0, \ldots, \epsilon_p)\big\rangle\big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q})\big\rangle \in \mathbb{Z}$
where $\Delta^p \xhookrightarrow{l(\epsilon_0, \ldots, \epsilon_p)} \Delta^{p+q} \xhookrightarrow{\quad T \quad} X$.

(Since g has moved $T \circ l(\epsilon_0, \ldots, \epsilon_p)$, the sign convention is in keeping with the convention of introducing a sign of $(-1)^{pq}$ whenever interchanging symbols of degree p and q.)

Notation: Let $1 \in S^0(X)$ be the element defined by $\langle 1, T\rangle = 1$ for all generators $T \in S_0(X)$. (Thus as a function in $\mathrm{Hom}\big(S_0(X), \mathbb{Z}\big) \cong \mathbb{Z}$, $1 = \epsilon =$ a generator.)

The following properties follow immediately from the definitions:

1. $f \cup (g + h) = (f \cup g) + (f \cup h)$
2. $(f + g) \cup h = (f \cup g) + (h \cup g)$
3. $(f \cup g) \cup h = f \cup (g \cup h)$
4. $1 \cup g = g \cup 1 = g$

So $\cup$ turns $S^*(X)$ into a ring (with unit). It is called a graded ring with $S^p(X)$ being the p gradation where:

Definition 9.10.1 *A ring R is called a graded ring if $\exists$ subgroups R_p s.t. $R = \oplus_p R_p$ and the multiplication satisfies $R_p \cdot R_q \subset R_{p+q}$.*

Lemma 9.10.2 *Let $f \in S^p(X)$ and $g \in S^q(X)$. Then $\delta(f \cup g) = \delta f \cup g + (-1)^p f \cup \delta g$.*

Proof: Let $T : \Delta^{p+q+1} \to X$ be a generator of $S_{p+q+1}(X)$.
$\langle \delta(f \cup g), T\rangle$
$\quad = (-1)^{(p+1)q}\big\langle \delta f, T \circ l(\epsilon_0, \ldots, \epsilon_p)\big\rangle\big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q+1})\big\rangle$
$\quad = (-1)^{(pq+q}(-1)^{p+1}\big\langle f, \partial T \circ l(\epsilon_0, \ldots, \epsilon_p)\big\rangle\big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q+1})\big\rangle$
$\quad = (-1)^{pq+p+q+1} \sum_{i=0}^{p+1} (-1)^i \big\langle f, T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i \ldots, \epsilon_p)\big\rangle\big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q+1})\big\rangle.$

Similarly

$(-1)^p f \cup \delta g$

$= (-1)^p(-1)^{pq+p+q+1} \sum_{i=p}^{p+q+1} (-1)^{i-p} \big\langle f, T \circ l(\epsilon_0, \ldots, \epsilon_p)\big\rangle \big\langle g, T \circ l(\epsilon_p, \ldots, \hat{\epsilon}_i \ldots, \epsilon_{p+q+1})\big\rangle$

$= (-1)^{pq+p+q+1} \sum_{i=p}^{p+q+1} (-1)^{i} \big\langle f, T \circ l(\epsilon_0, \ldots, \epsilon_p)\big\rangle \big\langle g, T \circ l(\epsilon_p, \ldots, \hat{\epsilon}_i \ldots, \epsilon_{p+q+1})\big\rangle$.

Notice that the term of $\langle \delta f \cup g, T\rangle$ corresponding to $i = p+1$ equals that of $(-1)^p \langle \delta(f \cup \delta g), T\rangle$ corresponding to $i = p$ except that the signs are opposite so they cancel when we form $\langle \delta f \cup g, T\rangle + (-1)^p \langle \delta(f \cup \delta g), T\rangle$. On the other hand,

$\langle \delta(f \cup g), T\rangle$

$= (-1)^{p+q+1} \langle f \cup \delta g, \partial T\rangle$

$$= (-1)^{p+q+1} \sum_{i=0}^{p+q+1} (-1)^i \big\langle f \cup g, T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i \ldots, \epsilon_{p+q+1})\big\rangle$$

$$= (-1)^{p+q+1}(-1)^{pq} \sum_{i=0}^{p} (-1)^i \big\langle f, T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i \ldots, \epsilon_{p+1})\big\rangle \big\langle g, T \circ l(\epsilon_{p+1}, \ldots, \hat{\epsilon}_i \ldots, \epsilon_{p+q+1})\big\rangle$$

$$= (-1)^{pq+p+q+1} \sum_{i=0}^{p} (-1)^i \big\langle f, T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i \ldots, \epsilon_{p+1})\big\rangle \big\langle g, T \circ l(\epsilon_{p+1}, \ldots, \hat{\epsilon}_i \ldots, \epsilon_{p+q+1})\big\rangle$$
$$+ (-1)^{pq+p+q+1} \sum_{i=p+1}^{p+q+1} (-1)^i \big\langle f, T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i \ldots, \epsilon_{p+1})\big\rangle \big\langle g, T \circ l(\epsilon_{p+1}, \ldots, \hat{\epsilon}_i \ldots, \epsilon_{p+q+1})\big\rangle$$

$= \langle \delta f \cup g + (-1)^p f \cup \delta g, T\rangle$. □

Corollary 9.10.3 *If $[f] \in H^p(X)$ and $[g] \in H^q(X)$ then $[f] \cup [g]$ is a well defined element of $H^{p+q}(X)$.*

Proof:

If $\delta f = 0$ and $\delta g = 0$ then $\delta(f \cup g) = 0$ by the lemma.

Also, if $f - f' = \delta h$ then $\delta(h \cup g) = \delta h \cup g + (-1)^{p+1} h \cup \delta g = (f - f') \cup g + 0 = f \cup g - f' \cup g$. Hence $[f \cup g] = [f' \cup g]$.

Similarly if $[g] = [g'] = \delta h$ then $[f \cup g] = [f \cup g']$. □

Proposition 9.10.4 $\delta 1 = 0$

Proof:

Let $T : I = \Delta_1 \to X$ be a generator of $S_1(X)$.

$\langle \delta 1, T\rangle = -\langle 1, \partial T\rangle = -\langle 1, T(1) - T(0)\rangle = -(1 - 1) = 0$ □

Corollary 9.10.5 *$H^*(X)$ is a graded ring with $[1]$ as unit.* $\square$

From now on we will write 1 for $[1] \in H^0(X)$.

Proposition 9.10.6 *Let $\phi : X \to Y$. Then $\phi^* : S^*(Y) \to S^*(X)$ and $\phi^* : H^*(Y) \to H^*(X)$ are ring homomorphisms.*

Proof:
$\langle \phi^*(f \cup g), T\rangle = \langle (f \cup g), \phi_* T\rangle = (-1)^{pq} \big\langle f, \phi_* T \circ l(\epsilon_0 \ldots, \epsilon_p)\big\rangle \big\langle g, \phi_* T \circ l(\epsilon_p \ldots, \epsilon_{p+q})\big\rangle = (-1)^{pq} \big\langle \phi^* f, T \circ l(\epsilon_0 \ldots, \epsilon_p)\big\rangle \big\langle \phi^* g, T \circ l(\epsilon_p \ldots, \epsilon_{p+q})\big\rangle \langle \phi^*(f) \cup \phi^*(g), T\rangle$ $\square$

Definition 9.10.7 *A graded ring $R = \oplus_p R_p$ is called* graded commutative *if for $a \in R_p$, $b \in R_q$, $ab = (-1)^{pq} ba$.*

Theorem 9.10.8 *$H^*(X)$ is graded commutative.*

Remark 9.10.9 *It is note true that $S^*(X)$ is grade commutative. Instead, $ab - (-1)^{pq} ba = \delta$(something).*

Proof:
Define $\theta : S_*(X) \to S_*(X)$ as follows. For a generator $T : \Delta^p \to X \in S_p(X)$ define $\theta(T) = (-1)^{\frac{1}{2}p(p+1)} T \circ l(\epsilon_p, \epsilon_{p-1}, \ldots, \epsilon_1, \epsilon_0) \in S_p(X)$.
Write $\lambda_p := (-1)^{\frac{1}{2}p(p+1)}$.

Lemma 9.10.10 *θ is a chain map.*

(The factor λ_p was included so that this would be true.)
Proof: For a generator $T \in S_p(X)$,
$\partial\theta(T) = \lambda_p \partial T \circ l(\epsilon_p, \ldots, \epsilon_0) = \lambda_p \sum_{i=0}^{p} (-1)^{p-i} T \circ l(\epsilon_p, \ldots, \hat{\epsilon_i}, \ldots, \epsilon_0)$.
$\theta\partial(T) = \theta\left(\sum_{i=0}^{p} (-1)^i T \circ l(\epsilon_0, \ldots, \hat{\epsilon_i}, \ldots, \epsilon_p)\right) = \lambda_{p-1} \sum_{i=0}^{p} (-1)^i T \circ l(\epsilon_p, \ldots, \hat{\epsilon_i}, \ldots, \epsilon_0)$.
However $\lambda_p (-1)^{p-i} = \lambda_p (-1)^{i-p} = (-1)^{\frac{1}{2}p(p+1)+i-p} = (-1)^{\frac{1}{2}(p^2-p)+i} = (-1)^i \lambda_p$. $\square$

Lemma 9.10.11 *$\theta \simeq i$*

Proof: Acyclic models.
If you examine the proof that sd $\simeq 1$ you discover that the only properties of sd use are:

1. $\forall f : X \to Y$, $f \circ \mathrm{sd}_X = \mathrm{sd}_Y \circ f$

2. $\mathrm{sd}_0 = 1 : S_0(X) \to S_0(X)$.

Since θ satisfies these also, the proof can be repeated, word for word, with θ replacing sd.

Proof of Theorem (cont.):

Since $\theta = \mathrm{id} : H_*(X) \to H_*(X,\ \theta^* = \mathrm{id} : H^*(X) \to H^*(X)$.

Let $[f] \in H^p(X)$, $[g] \in H^q(X)$. For a generator $T \in S_{p+q}(X)$:

$$\begin{aligned}
\langle \theta^*(f \cup g), T\rangle &= \langle (f \cup g), \theta T\rangle \\
&= \lambda_{p+q}\big\langle (f \cup g), \theta T \circ l(\epsilon_{p+q}, \ldots, \epsilon_0)\big\rangle \\
&= \lambda_{p+q}(-1)^{pq}\big\langle f, T \circ l(\epsilon_{p+q}, \ldots, \epsilon_q)\big\rangle\big\langle g, T \circ l(\epsilon_q, \ldots, \epsilon_0)\big\rangle \\
&= \lambda_{p+q}(-1)^{pq}\big\langle f, \lambda_p \theta T \circ l(\epsilon_q, \ldots, \epsilon_{p+q})\big\rangle\big\langle g, \lambda_q \theta T \circ l(\epsilon_0, \ldots, \epsilon_q)\big\rangle \\
&= \lambda_{p+q}\lambda_p\lambda_q(-1)^{pq}\big\langle \theta^* f, \theta T \circ l(\epsilon_q, \ldots, \epsilon_{p+q})\big\rangle\big\langle \theta^* g, T \circ l(\epsilon_0, \ldots, \epsilon_q)\big\rangle \\
&= \lambda_{p+q}\lambda_p\lambda_q\langle \theta^* f \cup \theta^* g, T\rangle
\end{aligned}$$

So $\theta^*(f \cup g) = \lambda_{p+q}\lambda_p\lambda_q \theta^* g \cup \theta^* f$.

Hence $[f \cup g] = [\theta^*(f \cup g)] = \lambda_{p+q}\lambda_p\lambda_q[\theta^* g] \cup [\theta^* f] = \lambda_{p+q}\lambda_q\lambda_q[g] \cup [f]$.

However

$$\begin{aligned}
\lambda_{p+q}\lambda_p\lambda_q &= (-1)^{\frac{1}{2}(p+q)(p+q+1)+\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)} \\
&= (-1)^{\frac{1}{2}(p^2+2pq+q^2+p+q+p^2+p+q^2+q)} \\
&= (-1)^{\frac{1}{2}(2p^2+2pq+2q^2+2p+2q)} \\
&= (-1)^{p^2+pq+q^2+p+q} \\
&= (-1)^{pq}(-1)^{p(p+1)}(-1)^{q(q+1)} = (-1)^{pq}.
\end{aligned}$$

□

This is a "real" sign: does not depend upon the sign conventions.

subsectionRelative Cup Products

Let $j : A \hookrightarrow X$.

$0 \to S_*(A) \xhookrightarrow{j_*} S_*(X) \xrightarrow{c_*} S_*(X, A) \to 0$.

$0 \to S^*(X, A) \xhookrightarrow{c^*} S^*(X) \xrightarrow{j^*} S^*(A) \to 0$.

Let $f \in S^p(X)$ and let $g \in S^q(X, A)$.

j^* is a ring homomorphism, so $S^*(X, A)$ is an ideal in $S^*(X)$. i.e. $f \cup c^* g \in S^{p+q}(X, A)$.

Write $f \cup g$ for $f \cup c^* g) \in S^{p+q}(X, A) \subset S^*(X)$. That is, $c^*(f \cup g) := f \cup c^* g$. (Explicitly, observe that $j^*(f \cup c^* g) = j^f \cup j^* c^* g = j^* f \cup 0 = 0$ so $f \cup c^* g \in \operatorname{Im} c^*$ and therefore it defines an element of S^{p+q} which we are writing as $f \cup g$.) In computer science language, we are "overloading" the symbol $\cup$, meaning that its interpretation depends upon its arguments.

Similarly if $f \in S^p(X, A)$ and '$g \in S^q(X)$ we can define an element of $S^{p+q}(X, A)$ denoted again $f \cup g$ by $c^*(f \cup g) := f \cup c^* g$.

If $\delta f = 0$ and $\delta g = 0$ then $c^*\delta(f \cup g) = \delta c^*(f \cup g) = \delta(f \cup c^g) = 0$, and so $\delta(f \cup g) = 0$ since c^* is a monomorphism. Therefore $[f] \cup [g] \in H^{p+q}(X, A)$.
Check that it this is well defined:

If $f - f' = \delta h$ then $c^*\delta(h \cup g) = \delta(h \cup c^*g) = \delta h \cup c^*g = f \cup c^*g - f' \cup c^*g = c*(f \cup g - f' \cup g)$. Therefore $\delta(h \cup g) = f \cup g - f' \cup g$ so $[f \cup g] = [f' \cup g]$. Also if $g - g' = \delta k$ then $c^*\delta(f \cup k) = \delta(f \cup c^*k) = \pm\big(f \cup c^*(g - g')\big) = \pm c^*(f \cup g - f \cup g')$. Hence $\delta(f \cup k) = \pm(f \cup g - f \cup g')$ so $[f \cup g] = [f \cup g']$ in $H'(X, A)$. Therefore $f \cup g$ is well defined.

Lemma 9.10.12 *Let $\phi : (X, A) \to (Y, B)$ be a map of pairs. Let $f \in S^p(Y)$ and let $g \in S^q(Y, B)$. Then $\phi^*(f \cup g) = (\phi^* f \cup \phi^* g) \in S^q(X, A)$.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^*(Y,B) & \xrightarrow{c_B^*} & S^*(Y) & \longrightarrow & S^*(B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} & & \\
0 & \longrightarrow & S^*(X,A) & \xrightarrow{c_A^*} & S^*(X) & \longrightarrow & S^*(A) & \longrightarrow & 0
\end{array}$$

$$\begin{aligned}
c_A^*\phi^*(f \cup g) &= \phi^* c_B^*(f \cup g) \\
&\overset{\text{(definition of rel. cup)}}{=} \phi^*(f \cup c_B^* g) \\
&\overset{(\phi^*\text{ring homom.})}{=} \phi^* f \cup \phi^* g \\
&= \phi^* f \cup c_A^* \phi^* g \\
&\overset{\text{(definition of rel. cup)}}{=} c_A^*(\phi^* f \cup \phi^* g)
\end{aligned}$$

Since c_A^* is a monomorphism. $\phi^*(f \cup g) = \phi^* f \cup \phi^* g)$. □

9.10.1 Cap Products

Given $g \in S^q(X)$ and $x \in S_{p+q}(X)$ define $g \cap x \in S_p(X)$ by $\langle f, g \cap x >:= \langle f \cup g, x\rangle$ for all $f \in S^p(X)$.
Note: This uniquely defines $g \cap x$ (if it defines it all; i.e. $\exists$ at most one element satisfying this definition) since:

Given an abelian group G, write G^* $\mathrm{Hom}(G, \mathbb{Z})$, If G is free abelian then the canonical map $G \to G^{**}$ is a monomorphism.
Proof: The corresponding statement for vector spaces is standard. Since G is free abelian, can choose a basis and repeat the vector space proof, or:

Let $V = G \otimes \mathbb{Q}$. Since G is free abelian the map $G \to V$ given by $g \mapsto g \otimes 1$ is a monomorphism so

$$\begin{array}{ccc} G & \rightarrowtail & V \\ \downarrow & & \downarrow \\ G^{**} & \longrightarrow & V^{**} \end{array}$$

shows $G \to G^{**}$ is a monomorphism.

Remark 9.10.13 *Even in the vector space case, $V \to V^{**}$ is not an isomorphism unless V is finite dimensional.*

Explicitly, for a generator $T : \Delta^{p+q} \to X$ of $S_{p+q}(X)$, the above "definition" for $g \cap x$ is becomes $g \cup T = (-1)^{pq} \big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q}) \big\rangle T \circ l(\epsilon_0, \ldots, \epsilon_p)$

(This formula shows that there does indeed exist an element satisfying the above definition.)
Proof: $\forall f \in S^p(X)$,

$$(-1)^{pq} \Big\langle f, \big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q}) \big\rangle, T \circ l(\epsilon_0, \ldots, \epsilon_p) \Big\rangle$$
$$= (-1)^{pq} \big\langle f, T \circ l(\epsilon_0, \ldots, \epsilon_p) \big\rangle \big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q}) \big\rangle = \langle f \cup g, T \rangle$$

Lemma 9.10.14 *If $g \in S^q(X)$, $x \in S_{p+q}(X)$ then $\partial(g \cap x) = \delta g \cap x + (-1)^q (g \cap \partial x)$.*

Proof: Given $f \in S^{p-1}(X)$,

$$\begin{aligned} \langle f, g \cap \partial x \rangle &= \langle f \cup g, \partial x \rangle \\ &= (-1)^{p+q} \langle \delta(f \cup g), x \rangle \\ &= (-1)^{p+q} \langle \delta f \cup g + (-1)^{p-1} f \cup \delta g), x \rangle \\ &= (-1)^{p+q} \langle \delta f \cup g \rangle + (-1)^{q-1} \langle f \cup \delta g), x \rangle \\ &= (-1)^{p+q} \langle \delta f, g \cap x \rangle + (-1)^{q-1} \langle f, \delta g \cap x \rangle \\ &= (-1)^{p+q} (-1)^p \langle f, \partial(g \cap x) \rangle + (-1)^{q-1} \langle f, \delta g \cap x. \rangle \end{aligned}$$

Therefore $g \cap \partial x = (-1)^{-q} \partial(g \cap x) \rangle + (-1)^{q-1} \delta g \cap x \rangle$ or equivalently $\partial(g \cap x) = \delta g \cap x + (-1)^q (g \cap \partial x)$. □

It follows that if $[g] \in H^q(X)$, $[x] \in H_{p+q}(X)$, then $[g] \cap [x]$ is an element of $H_p(X)$. (Proof that it is well defined left as an exercise.)

There are also two versions of a relative cap product:

Let $j : A \hookrightarrow X$.

$0 \to S_*(A) \xhookrightarrow{j_*} S_*(X) \xrightarrow{c_*} S_*(X, A) \to 0$.

$0 \to S^*(X, A) \xhookrightarrow{c^*} S^*(X) \xrightarrow{j^*} S^*(A) \to 0$.

Let $g \in S^q(X)$ and let $x \in S^{p+q}(X, A)$.

Define $g \cap x \in S_p(X, A)$ by $\langle f, g \cap x \rangle = f \cup g, x >$ for $f \in S^p(X, A)$ (where $f \cup g$ is the relative cup product).
Or: If $g \in S^q(X, A)$, $x \in S_{p+q}(X, A)$ can define $g \cap x \in S_p(X)$ by $\langle f, g \cap x \rangle = f \cup g, x >$ for $f \in S^p(X)$ (where again $f \cup g$ is the relative cup product).

In each case, whenever g and x represent homology classes, $[g] \cap [x]$ is a well defined homology class of $H_p(X, A)$ or $H_p(X)$ respectively. (Exercise)

Lemma 9.10.15 *Let $\phi : (X, A) \to (Y, B)$. Let $g \in S^q(Y, B)$ and let $x \in S_{p+q}(X, A)$. Then $\phi_*(\phi^* g \cap x) = g \cap \phi_* x$ in $S_p(Y)$.*

Proof: Let $\in S^p(Y)$. Then

$$
\begin{aligned}
&\langle f, \phi_*(\phi^* g \cap x) \rangle \\
&\quad = \langle \phi^* f, \phi^* g \cap x \rangle \\
&\quad = \langle \phi^* f \cup \phi^* g, x \rangle \qquad \text{(where } \cup \text{ is the relatively cup product)} \\
&\quad \overset{\text{(lemma 9.10.12)}}{=} \langle \phi^*(f \cup g), x \rangle \\
&\quad = \langle f \cup g, \phi_* x \rangle \\
&\quad = \langle f, g \cap \phi_* x \rangle
\end{aligned}
$$

so $\phi_*(\phi^* g \cap x) = g \cap \phi_* x$. □

Lemma 9.10.16 *Suppose $Y \subset X$. Suppose $Y = Y_1 \cup Y_2$ and $X = X_1 \cup X_2$ where Y_ϵ and X_ϵ are open in X. Let $A = X_1 \cap X_2$, $B = Y_1 \cap Y_2$. Suppose also that $X_\epsilon \cup Y_\epsilon = X$ for $\epsilon = 1, 2$. Let $[v] \in H_n(X, B)$. Then the following diagram commutes $\forall q \leq n$:*

$$
\begin{array}{ccc}
H^{q-1}(X, B) & \xrightarrow{\Delta^*} & H^q(X, Y) \\
 & & \downarrow \cong \text{(excision)} \\
\downarrow \cap[v] & & H^q(A, A \cap Y) \\
 & & \downarrow \cap[v'] \\
H_{n-q+1}(X) & \xrightarrow{\Delta_*} & H_{n-q}(A)
\end{array}
$$

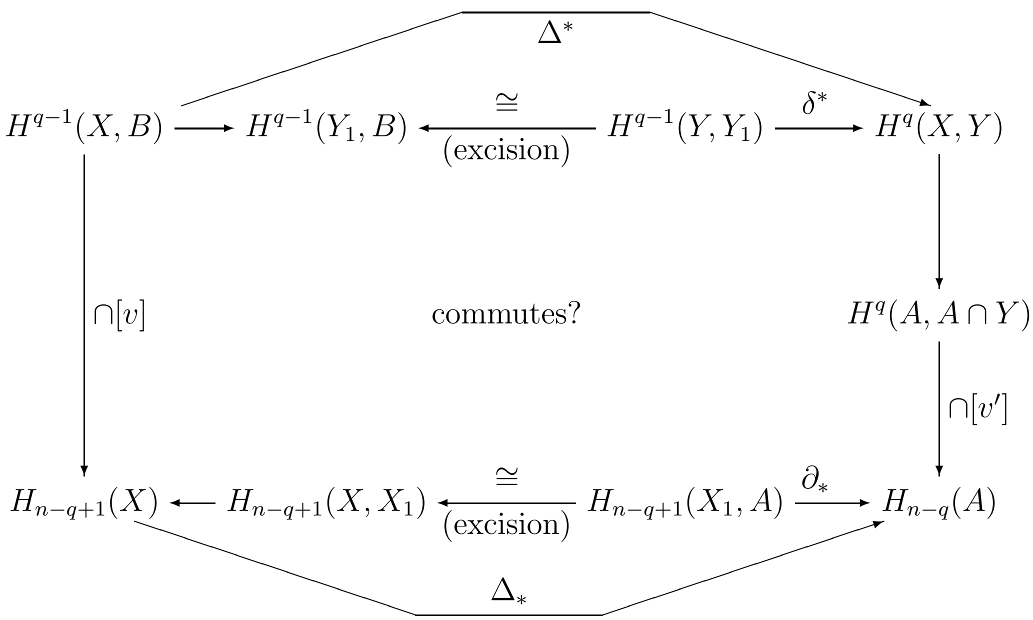
where:

$$\begin{array}{ccccc} [v] & & & & [v'] \\ H_n(X,B) \longrightarrow & H_n(X,Y) & \xleftarrow[\text{(excision)}]{\cong} & H_n(A, A\cap Y) & \end{array}$$

defines [v'] and Δ_* *and* Δ^* *are the connectiing homomorphisms from the Mayer-Vietoris sequences*

$$\ldots H_{n-q+1}(X) \xrightarrow{\Delta_*} H_{n-q}(A) \to H_{n-q}(X_1) \oplus H_{n-q}(X_2) \to H_{n-q}(X) \xrightarrow{\Delta_*} \ldots$$
$$\ldots H^{q-1}(X,B) \xrightarrow{\Delta^*} H^q(X,Y) \to H^q(X,Y_1) \oplus H^q(X,Y_2) \to H^q(X,B) \xrightarrow{\Delta_*} \ldots$$

Proof: By definition of Δ_* and Δ^* they factor as show below:



where ∂_* and ∂^* are connecting maps from long exact sequences.

The open sets $\{X_1 \cap Y_2, X_2 \cap Y_1, A\}$ cover X because:

$$\begin{aligned} (X_1 \cap Y_2) \cup (X_2 \cap Y_1) \cup A &= (X_1 \cap Y_2) \cup (X_2 \cap Y_1) \cup (X_1 \cap X_2) \\ &= (X_1 \cap Y_2) \cup \big(X_2 \cap (Y_1 \cup X_1)\big) \\ &= (X_1 \cap Y_2) \cup X_2 \\ &= (X_1 \cup X_2) \cap (Y_2 \cup X_2) = X \cap X = X \end{aligned}$$

Therefore by corollary 9.4.12 (used in the proof of excision,) $[v]$ has a representative $u \in S_n(X)$ where $u = u_1 + u_2 + u'$ with $u_1 \in S_n(X_1 \cap Y_2)$, $u_2 \in S_n(X_2 \cap Y_1)$, $u' \in S_n(A)$, and $\partial u \in S_{n-1}(B)$.

That is, by corollary 9.4.12, $S_*^{\mathcal{A}}(X,B) \to S_*(X,B)$ induces an isomorphism on homology, where $\mathcal{A} = \{X_1 \cap Y_2, X_2 \cap Y_1, A\}$. Therefore $\exists$ a representative $\tilde{u}$ of $[v]$ lying in $S_n^{\mathcal{A}}(X,B)$ which means that if we take a preimage u of $\tilde{u}$ back in $S_n^{\mathcal{A}}(X)$ then $u = u_1 + u_2 + u'$ as above with $\partial y \in S_{n-1}(B)$.

Notice that since $u_1, u_2 \in S_n(Y)$, then their images in $S_n(X,Y)$ vanish so that the image of $[v]$ under $H_n(X,B) \to H_n(X,Y)$ is represented by the reduction of $u' \mod S_n(Y)$. Hence $[v'] = [u'] \mod S_n(Y)$.

Left-bottom image of $[f] \in H^{q-1}(X,B)$ is

$$\Delta_*(f \cap u) = \Delta_*(f \cap u_1] + \Delta_*(f \cap u_2] + \Delta_*(f \cap u'].$$

However $U_2 \in S_n(X_2 \cup Y_1) \subset S_n(X_2)$ and $u' \in S_n(A) \subset S_n(X_2)$.

Therefore $f \cap u_2 \in S_{n-q+1}(X_2)$ and $f \cup u' \in S_{n-q+1}(X_2)$. (More precisely, if $j_2 : X_2 \hookrightarrow X$ then $j_{2*}(j_2^* f \cap u_2) = f \cap j_{2*} u_2 = f \cap u_2$, identifying u_2 with its image under the monomorphism j_{2*}. So $f \cap u_2 \in \operatorname{Im} j_{2*}$.)

Hence $f \cap u_2$ and $f \cap u'$ die under the map $S_{n-1+1}(X) \to S_{n-q+1}(X, X_2)$, (which is part of Δ_*) and thus $\Delta_*[f \cap u] = \Delta_*[f \cap u_1]$.

Notice that $\Delta_*[f \cap u_1] = \partial[f \cap u_1]$ because as above $f \cap u_1 \in S_*(X_1)$ and so its reduction mod $S_*(A)$ gives the image under the excision isomorphism and thus *it* serves as a suitable pre-image of the reduction to be used when computing the connecting homomorphism ∂.

Finally, $\partial[f \cap u_1] = [\partial f \cap u_1] + (-1)^{q-1}[f \cap \partial u_1] = (-1)^{q-1}[f \cap \partial u_1]$, since f is a cocycle.

To summarize, the left-bottom image of $[f]$ is $(-1)^{q-1}[f \cap \partial u_1]$

To compute the other way around the figure:

The image of $[f]$ under $H^{q-1}(X,B) \to H^{q-1}(Y_2,B)$ is represented by the restriction of f to $S_{q-1}(Y_2)$. The image under the excision isomorphism is represented by a cocycle $f' \in S^{q-1}(Y,Y_1)$ whose restriction to Y_2 is homologous to $f\big|_{S_{q-1}(Y_2)}$ within $S^{q-1}(Y_2,B)$. That is, $\exists \in S^{q-2}(Y_2,B)$ s.t. $f'\big|_{S_{q-1}}(Y_2) = f\big|_{S_{q-1}}(Y_2) + \delta g$.

We modify f' so as to eliminate δg as follows:

$g \in S^{q-2}(Y_2,B)$ is defined on $S_{q-2}(Y_2)$. Extend it to a g' defined on $S_{q-2}(Y_2)$ by defining it to be zero on all generators of $S_{q-2}(Y)$ lying outside $S_{q-2}(Y_2)$. (We are using, in effect, that $S_{q-2}(Y_2) \hookrightarrow S_{q-2}(Y)$ splits.) Let $f'' = f' - \delta g' \in S^{q-1}(Y)$. Then f'' is still a cocycle, $[f''] = [f']$ and $f''\big|_{S_{q-1}}(Y_2) = f\big|_{S_{q-1}}(Y_2)$. Extend f'' to an element $\tilde{f} \in S^{q-1}(X)$ (for example, by setting it to be zero on generators outside $S_{q-1}(Y)$. Note: $\tilde{f}$ need no longer be a cocycle.) $\tilde{f}$ is thus a pre-image of f'' under the surjection $S^{q-1}(X,Y_1) \twoheadrightarrow S^{q-1}(Y,Y_2)$ and so is a suitable element for computing $\delta^*[f'']$. That is $\delta^*[f''] = [\delta\tilde{f}]$. (It needn't be the 0 homology class because $\tilde{f} \notin S^q(X,Y)$: it isn't zero on $S_*(Y)$.) So $\Delta^*[f] = [\delta\tilde{f}]$.

Thus the top-right image of $[f]$ is $[\delta\tilde{f}] \cap [v'] = [\delta\tilde{f} \cap u']$ (where, more precisely, we should write the restriction of $\delta\tilde{f}$ to $S_*(A)$ rather than $\delta\tilde{f}$.)

Since $u' \in S_*(A)$, $\tilde{f} \cap u' \in S_*(A)$, so $[\partial(\tilde{f} \cap u')] = 0$ in $S_{n-q}(A)$.

$\partial(\tilde{f} \cap u') = \delta\tilde{f} \cap u' + (-1)^{q-1}\tilde{f} \cap \partial u'$ so $[\partial(\tilde{f} \cap u')] = -(-1)^{q-1}[\tilde{f} \cap \partial u']$.

Therefore it remains to show that $[\tilde{f} \cap \partial u'] = -[f \cap \partial u_1]$.

However $\tilde{f} \cap \partial u' = \tilde{f} \cap \partial u - \tilde{f} \cap \partial u_1 - \tilde{f} \cap \partial u_2$.

$\partial u \in S_{n-1}(B) \subset S_{n-1}(Y_1)$ and $u_2 \in S_n(X_2 \cap Y_1) \subset S_{n-1}(Y_1)$ and so $\partial u_1 \in S_{n-1}(Y_2)$.

Similarly $\partial U_1 \in S_{n-1}(Y_2)$.

But $\tilde{f}\big|_{S_*(Y)} = f''\big|_{S_*(Y)}$ and $\tilde{f}\big|_{S_*(Y_2)} = f''\big|_{S_*(Y_2)} = f\big|_{S_*(Y_2)}$. Hence $\tilde{f} \cap \partial u = f'' \cap \partial u$, $\tilde{f} \cap \partial u_2 = f'' \cap \partial u_2$, $\tilde{f} \cap \partial u_1 = f'' \cap \partial u_2$.

The first two terms are zero, since $f''\big|_{Y_1} = 0$. Thus $[\tilde{f} \cap \partial u'] = -[f \cap \partial u_1]$, as desired. □