Chapter 8

Covering Spaces

8.1 Introduction to covering spaces

Covering spaces have many uses both in topology and elsewhere. Our immediate goal is to use them to help compute $\pi_1(X)$.

Definition 8.1.1 *A map $p : E \to X$ is called a* covering projection *if every point $x \in X$ has an open neighbourhood U_x s.t. $p^{-1}(U_x)$ is a (nonempty) disjoint union of open sets each of which is homeomorphic by p to U_x. E is called the* covering space*, X the* base space *of the covering projection.*

Remark: It is clear from the definition that a covering projection must be onto.

Example: $\mathbb{R} \overset{\exp}{\to} S^1$ by $t \mapsto e^{2\pi it}$
$\exp^{-1}(U_x) = \coprod_{n=-\infty}^{\infty} V_n$.
$V_n \cong U_x \forall n$.

More generally: A (left) action of a topological group G on a topological space X consists of a (continuous) map $\phi : G \times X \to X$ s.t.

1. $ex = x \ \forall x$

2. $g_1(g_2x) = (g_1g_2)x \ \forall g_1, g_2 \in G, x \in X$.

Given action $\phi : G \times X \to X$, for each $g \in G$ we get a continuous map $\phi_g : X \to X$ sending x to gx. Each ϕ_g is a homeomorphism since $\phi_{g^{-1}} = (\phi_g)^{-1}$.

Note: Any group becomes a topological group if given the discrete topology. In the case where G has the discrete topology, ϕ is continuous $\Leftrightarrow$ ϕ_g is continuous $\forall g \in G$. (In general, ϕ_g continuous for all g is not sufficient to conclude that ϕ is continuous.)

Suppose G acts on X.

Define an equivalence relation on X by $x \sim gx\ \forall x \in X$, $g \in G$. Write X/G for $X/\sim$ (with the quotient topology).

Remark: The notation is in conflict with the previously given notation that X/A means identify the points of A to a single point. Rely on context to decide which is meant.

Preceding example: $X = \mathbb{R}$, $G = \mathbb{Z}$. $\phi(n, x) = x + n$. Then $\mathbb{R}/\mathbb{Z} \cong S^1$. In this example X happens to also be a topological group and G a normal subgroup so X/G also has a group structure. The homeomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$ is an isomorphism of topological groups.

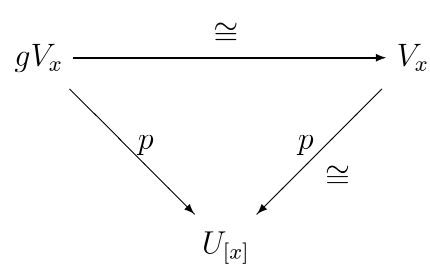
Theorem 8.1.2 *Suppose a group G acts on a space X s.t. $\forall x \in X$, $\exists$ an open neighbourhood V_x s.t. $V_x \cap gV_x = \emptyset$ for all $g \neq e$ in G. Then the quotient map $p : X \to X/G$ is a covering projection.*

Proof: Given $[x] \in X/G$, find V_x as in the hypothesis. Set $U_{[x]} = p(V_x)$. $p^{-1}(U_{[x]}) = \bigcup_{g \in G} g \cdot V_x$.

V_x open $\Rightarrow gV_x$ open $\forall g \Rightarrow p^{-1}(U_{[x]})$ open $\Rightarrow U_{[x]}$ open.

$g_1 V_x \cap g_2 V_x = \emptyset$ so the union is a disjoint union.

$p : V_x \to U_{[x]}$ is a bijection and check that by definition of the quotient topology it is a homeomorphism.



Both gV_x and V_x map to $U_{[x]}$ under p, and the map p composed with $g : V_x \to gV_x$ equals the map $p : V_x \to U_{[x]}$, which shows that $p|_{gV_x}$ is a homeomorphism $\forall g$.

Hence $p : X \to X/G$ is a covering projection.

Corollary 8.1.3 *Suppose H is a topological group and G a closed subgroup of H s.t. as a subspace of H, G has the discrete topology Then $p : H \to H/G$ is a covering projection.*

Example 2: $S^n \to \mathbb{R}P^n$ is a covering projection.

Proof: $\mathbb{R}P^n = S^n/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \{-1, 1\}$ acts by $1x = x, -1x = -x$. Furthermore, the hypothesis of the previous theorem is satisfied.

Similarly $\mathbb{C}P^n = S^{2n+1}/S^1$ and $\mathbb{H}P^n = S^{4n+3}/SU(2)$, but these quotient maps are not covering projections (since the group is not discrete).

What have covering spaces got to do with $\pi_1(X)$?

Return to the example $\mathbb{R} \overset{\exp}{\to} S^1$.

Let w be a path in $\mathbb{R}$ which begins at 0 and ends at the integer n. w is not a closed curve in $\mathbb{R}$ (unless $n = 0$, where in this context "closed" means a curve which ends at the point at which it starts) but $\exp(w)$ is a closed curve in S^1 joining $*$ to $*$.

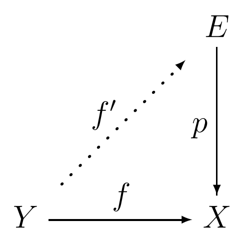
So $\exp(w)$ represents an element of $\pi_1(S^1)$.

We will show that the resulting element of $\pi_1(S^1)$ depends only on n (not on w) and that this correspondence sets up an isomorphism $\pi_1(S^1) \cong \mathbb{Z}$.

Terminology: Let $p : E \to X$ be a covering projection. Let $U \subset X$ be open. If $p^{-1}(U)$ is a disjoint union of open sets each homeomorphic to U, then we say that U is *evenly covered.* If $U \subset X$ is evenly covered, with $p^{-1}(U) = \coprod_i T_i$ with $T_i \cong U$, then each T_i is called a *sheet* over U.

Theorem 8.1.4 (Unique Lifting Theorem) *Let $p : (E, e_0) \to (X, x_0)$ be a map of pointed spaces in which $p : E \to X$ is a covering projection.*

Let $f : (Y, y_0) \to (X, x_0)$. If Y is connected, then there is at most one map $f' : (Y, y_0) \to (E, e_0)$ s.t.



Remark 8.1.5 *: For this theorem it suffices to know that Y is connected under the standard definition, although in most applications we will actually know that Y is path connected, which is even stronger.*

Proof:

Suppose $f', f''(Y, y_0) \to (E, e_0)$ s.t. $pf' = f$ and $pf'' = f$. Let $A = \{y \in Y \mid f'(y) = f''(y)\}$, $B = \{y \in Y \mid f'(y) \neq f''(y)\}$. Then $A \cap B = \emptyset$, $A \cup B = Y$.

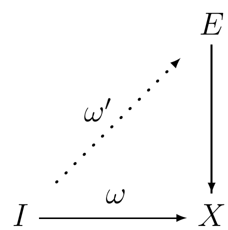
It suffices to show that both A and B are open because then one of them is empty. But $A \neq \emptyset$ since $y_0 \in A$, so this would imply that $B = \emptyset$ and $A = X$, in other words $f' = f''$.

To show A is open: Let $y \in A$. Let U be an evenly covered set in X containing $f(y)$. Let S be a sheet in $p^{-1}(U)$ containing $f'(y) = f''(y)$. Let $V = (f')^{-1}(S) \cap (f'')^{-1}(S)$, which is open in Y and contains y. $\forall v \in V$, $pf'(v) = f(v) = pf''(v) \Rightarrow f'(v) = f''(v)$ (since $p|_S$ is a homeomorphism). Hence $V \subset A$, so y is interior. So A is open.

To show B is open: Let $y \in B$. Let U be an evenly covered set containing $f(y)$. $f'(y) \neq f''(y)$ but $pf'(y) = f(y) = f''(y)$ so $f'(y)$ and $f''(y)$ lie in different sheets (say S', S'') over $p^{-1}(U)$.

Let $V = (f')^{-1}(S') \cap (f'')^{-1}(S'')$, which is open in Y. Since $S' \cap S = \emptyset$, $f'(V) \neq f''(V)$ $\forall v \in V$. Hence $V \subset B$. So y is interior. Therefore B is open. $\square$

Theorem 8.1.6 *(***Path Lifting Theorem***) Let* $(E, e_0) \xrightarrow{p} (X, x_0)$ *be a covering projection. Let* $w : I \to X$ *s.t.* $w(0) = x_0$*. Then* w *lifts uniquely to a path* $w' : I \to E$ *s.t.* $w'(0) = e_0$*.*



Proof: Uniqueness follows from the previous theorem (since I is connected).

Existence: Cover X by evenly covered sets. Using a Lebesgue number for the inverse images under w in the compact set I, we can partition I into a finite number of subintervals $[t_i, t_{i+1}]$ $(0 = t_0 < t_1 < \cdots < t_n = 1)$ s.t. $\forall i$, $w([t_i, t_{i+1}]) \subset U_i$. Note that U_i is evenly covered.

Let S_0 = sheet in $p^{-1}(U_0)$ containing e_0. $p|_{S_0}$ is a homeomorphism $\Rightarrow \exists$ unique path in S_0 covering $w([t_0, t_1])$. Let e_1 denote the end of this path. $(p(e_1) = w(t_1))$

Let S_1 = sheet in $p^{-1}(U_1)$ containing e_1.

As above, $\exists$ unique path in S_1 covering $w([t_1, t_2])$.

Continuing: Build a path w' in E beginning at e_0 and covering w. $\square$

Remark 8.1.7 *The procedure is reminiscent of analytic continuation. Notice that even through* ω *is closed* $(\omega(0) = \omega(1))$*, this need not be true for* ω'*. e.g. Consider* $p = \exp : \mathbb{R} \to S^1$ *and let* $\omega(t) = e^{2\pi\iota t} : I \to S^1$*. Then* ω' *is the line segment joining* 0 *to* 1*.*

We will show that under the right conditions (e.g. $\mathbb{R} \to S^1$) elements of $\pi_1(X, x_0)$ can be identified by the endpoint in E of the lifted representing path.

Need:

Theorem 8.1.8 *(Covering Homotopy Theorem) Let* $p : (E, e_0) \to (X, x_0)$ *be a covering projection. Let* (Y, y_0) *be a pointed space. Let* $f : (Y, y_0) \to (X, x_0)$ *and let* $f' : (Y, y_0) \to (E, e_0)$ *be a lift of* f*. Let* $H : Y \times I \to X$ *be a homotopy with* $H - 0 = f$*. Then* H *lifts to a homotopy* $H' : Y \times I \to E$ *s.t.* $H'_0 = f'$*.*

Before the proof, we examine the consequences.

Corollary 8.1.9 *Let* $(E, e_0) \to (X, x_0)$ *be a covering projection. Let* $\sigma, \tau : I \to X$ *be paths from* x_0 *to* x_1 *s.t.* $\sigma \simeq \tau \operatorname{rel}\{0,1\}$*. Let* σ'*,* τ' *be lifts of* σ*,* τ *respectively, beginning at* e_0*. Then* $\sigma'(1) = \tau'(1)$ *and* $\sigma' = \tau' \operatorname{rel}\{0,1\}$*.*

Note in particular that this implies that the endpoint of a lift of a homotopy class is independent of the choice of representative for that class.

Proof of Corollary (assuming Theorem): Let $H : \sigma \simeq \tau \operatorname{rel}\{0,1\}$. Apply the theorem to get $H' : I \times I \to E$ which lifts H and s.t. $H'_0 = \sigma'$

The left vertical line of H' can be thought of as a path in E begining at $\sigma'(0) = e_0$ and lifting c_{x_0}. By uniqueness it must be c_{e_0}. Similarly the right must be c_{e_1}, where $e_1 = \sigma'(1)$. Also, the top is a lift of τ beginning at e_0 so it must be τ'. Thus $H' : \sigma' \simeq \tau' \operatorname{rel}\{0,1\}$ and $\tau'(1) =$ upper right corner $= e_1 = \sigma'(1)$. □

Proof of Theorem:
Technical remark: It is easy to define the required lift, but not so easy to show continuity. i.e. Given $y \in I$, $H\big|_{y\times I}$ is a path in X beginning at $f(y)$ so $H'\big|_{f'(y)\times I}$ is the unique lift beginning at $f'(y)$.

Step 1: $\forall y \in Y$, $\exists$ open neighbourhood V_y and a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of I (depending on y) s.t. $\forall i$, $H(V_y \times [t_i, t_{i+1}])$ is contained in an evenly covered set.
Proof: Given y:

$\forall t \in I$ find evenly covered neighbourhood U_t of $H(y,t)$ in X.

Find basic open $A_t \times B_t \subset H^{-1}(U_t) \subset Y \times I$ containing (y,t). Then $\cup_{t\in I} B_t$ covers I so choose a finite subcover $B_{t_1}, \ldots B_{t_{n-1}}$. Set $V_y := A_{t_1} \cap \cdots \cap A_{t_{n-1}} \cap A_0 \cap A_1$. Use V_y together with the partition $0 < t_1 < \ldots < t_{n-1} < 1$. √

Step 2: $\forall y$, $\exists$ continuous $H'_y : V_y \times I \to E$ lifting $H\big|_{V_y\times I}$ and extending $H'_y\big|_{V_y\times 0} = f'\big|_{V_y}$.
Proof: Use the same inductive argument as in the proof of the Path Lifting Theorem. √

Step 3: The various lifings H'_y from Step 2 combine to produce a well defined map of sets $H' : Y \times I \to E$.
Proof: Suppose $(y,t) \in (V_{t_1} \times I) \cap (V_{y_2} \times I)$. The restrictions $H'_{y_1}\big|_{y\times I}$ and $H'_{y_2}\big|_{y\times I}$ each produce paths in E beginning at $f'(y)$ and lifting $H\big|_{y\times I}$. So by unique path lifting, $H'_{y_1}(y,t) = H'_{y_2}(y,t)$. Hence the value of $H'(y,t)$ is independent of the set V_{y_i} used to compute it. i.e. H' is well defined. √

Step 4: The map H' defined in Step 3 is continuous.
Proof: Suppose $U \subset E$ is open.

$H'^{-1}(U) = \bigcup_{y\in U} (H'_y)^{-1}(U)$.

$\forall y \in U$, $H'_y : V_y \times I \to E$ is continuous which implies that $(H'_y)^{-1}(U)$ is open in $V_y \times I$. Since $V_y \times I$ is open in $Y \times I$, this implies that $(H'_y)^{-1}(U)$ is open in ${H'}^{-1}(U)$. Hence ${H'}^{-1}(U)$ is open and thus H' is continuous. □

Corollary 8.1.10 *Let $p : (E, e_0) \to (X, x_0)$ be a covering projection. Then $p_\# : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is a monomorphism.*

Proof: Let $[\omega] \in \pi_1(E, e_0)$. ω is a path in E beginning and ending at e_0. Suppose $p_\#([\omega]) = 1$. Then $p \circ \omega \simeq c_{x_0} \operatorname{rel}\{0,1\}$. By the Corollary 8.1.9, $(p \circ \omega)' \simeq c'_{x_0} \operatorname{rel}\{0,1\}$ where $(p \circ \omega)'$, c'_{x_0} are, respectively, the lifts of $p \circ \omega$, c_{x_0} beginning from e_0. Clearly these lifts are ω and c_{e_0} respectively. Hence $\omega \simeq c_{e_0} \operatorname{rel}\{0,1\}$, so $[\omega] = 1 \in \pi_1(E, e_0)$. □

Theorem 8.1.11 $\pi_1(S^1) \cong \mathbb{Z}$

Proof: Let $\omega : (S^1, *) \to (S^1, *)$ represent an element of $\pi_1(S^1, *)$. Regard ω as a path which begins and ends at $*$. By unique path lifting in $\exp : (\mathbb{R}, 0) \to (S^1, *)$ we get a path ω' in $\mathbb{R}$ lifting ω beginning at 0. Hence $\exp\big(\omega'(1)\big) = \omega(1) = *$ so $\omega'(1) = n \in \mathbb{Z}$. By Corollary 8.1.9 n is independent of the choice of representative for the class $[\omega]$. Thus we get a well defined $\phi : \pi_1(S^1) \to \mathbb{Z}$ given by $[\omega] \mapsto \omega'(1)$.

Claim: ϕ is a group homomorphism.

Let $\sigma, \tau : (S^1, *) \to (S^1, *)$ represent elements of $\pi_1(S^1)$. Let $\sigma', \tau' : I \to \mathbb{R}$ be lifts of σ, τ respectively beginning at 0. Let $n = \sigma'(1) = \phi([\sigma])$ and $m = \tau'(1) = \phi([\tau])$. Define τ'' by $\tau''(t) = \tau'(t) + n$. Then τ'' = lift of τ beginning at n, ending at $n+m$. The path $\sigma' \cdot \tau''$ in $\mathbb{R}$ makes sense (since $\sigma'(1) = n = \tau''(0)$). $\sigma' \cdot \tau''$ begins at 0 and ends at $n+m$. But $\exp(\sigma' \cdot \tau'') = \sigma \cdot \tau$ so it lifts $\sigma \cdot \tau$. Hence $\phi([\sigma][\tau]) = \phi([\sigma \cdot \tau]) = n + m = \phi([\sigma]) + \phi([\tau])$. Thus ϕ is a homomorphism. √

Claim: ϕ is injective

Suppose $\phi([\sigma]) = 0$. Let $\sigma' : I \to \mathbb{R}$ be the lift of σ beginning at 0. Then the definition of ϕ implies that σ' ends at 0 so σ' represents an element of $\pi_1(\mathbb{R})$ and $\exp_\#([\sigma']) = [\sigma]$. But $\mathbb{R}$ is simply connected ($\pi_1(\mathbb{R}) = 1$) and so $[\sigma'] = 1$ which implies $[\sigma] = 1$. √

Claim: ϕ is onto

Given $n \in \mathbb{Z}$, let ω' be any path in $\mathbb{R}$ joining 0 to n. Let $\omega = \exp \circ\, \omega' : I \to S^1$. Then ω is a closed path in S^1 and $\phi([\omega]) = n$. □

Corollary 8.1.12 $\pi_1(\mathbb{C} - \{0\}) \cong \mathbb{Z}$

Proof: $S^1 \to \mathbb{C} - \{0\}$ is a homotopy equivalence. □

We wish to apply the method used above to calculate $\pi_1(S^1)$ to calculate $\pi_1(X)$ for other spaces X. For this, we need a covering projection $E \to X$, called the universal covering projection of X with properties described in the next section. For reference, we note here the properties of $\mathbb{R} \to S^1$ which were needed in the calculation of $\pi_1(S^1)$.

1. $\mathbb{Z}$ acts on $\mathbb{R}$, $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, by $(n, x) \mapsto n + x$ s.t.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\quad T_n \quad} & \mathbb{R} \\ & {\scriptstyle \exp} \searrow \qquad \swarrow {\scriptstyle \exp} & \\ & S^1 & \end{array}$$

   where T_n is the translation $T_n(X) = n + x$.

2. $\pi_1(\mathbb{R}) = 1$

We will return to this later. First some applications.

Theorem 8.1.13 *$\nexists f : D^2 \to S^1$ s.t.*

$$\begin{array}{ccc} S^1 & \hookrightarrow & D^2 \\ & {\scriptstyle 1_{S^1}} \searrow \qquad \swarrow {\scriptstyle f} & \\ & S^1 & \end{array}$$

commutes.

Proof: If f exists then, since D^2 is contractible, applying π_1 yields

$$\begin{array}{ccc} \mathbb{Z} = \pi_1(S^1) & \hookrightarrow & \pi_1(D^2) = 0 \\ & {\scriptstyle 1} \searrow \qquad \swarrow {\scriptstyle f_\#} & \\ & \mathbb{Z} = \pi_1(S^1) & \end{array}$$

This is a contradiction so f does not exist. $\square$

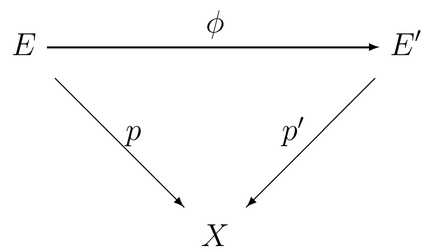
Corollary 8.1.14 *(Brouwer Fixed Point Theorem): Let* $g : D^2 \to D^2$. *Then* $\exists x \in D^2$ *such that* $g(x) = x$.

Proof: Suppose g has no fixed point. Define $f : D^2 \to S^1$ as follows:
$g(x) \neq x$ implies that $\exists$ a well defined line segment joining $g(x)$ to x. Follow this line until it reaches S^1 and call this point $f(x)$.

f is a continuous function of x (since g is) and if $x \in S^1$ then $f(x) = x$. This contradicts the previous theorem. Hence g has no fixed point. □

8.2 Universal Covering Spaces

Definition 8.2.1 *Let $p : E \to X$ and $p' : E' : X$ be covering projections. A* morphism *of covering spaces over X consists of a map $\phi : E \to E'$ s.t.*



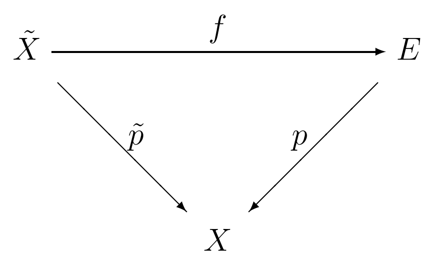
commutes.

A morphism of covering spaces which is also a homeomorphism is called an equivalence of covering spaces.

Remark: Covering spaces over a fixed X together with this notion of morphism form a category. An equivalence is an isomorphism in this category.

Definition 8.2.2 *A covering projection $\tilde{p} : \tilde{X} \to X$ is called the* universal covering projection *of X (and $\tilde{X}$ is called the* universal covering space *of X) if for any covering projection $p : E \to X$ $\exists!$ morphism $f : \tilde{X} \to E$ of covering projections.*

i.e.



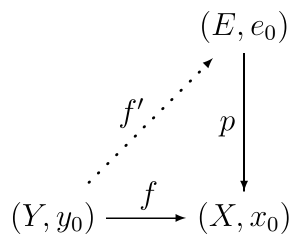
commutes.

Remark: This says $\tilde{p} : \tilde{X} \to X$ is an initial object in the category of covering spaces over X.

Proposition 8.2.3 *If X has a universal covering space then it is unique up to equivalence of covering spaces.*

Proof: Standard categorical argument. □

Theorem 8.2.4 *(***Lifting Theorem***) Let $p : (E, e_0) \to (X, x_0)$ be a covering projection and let $f : (Y, y_0) \to (X, x_0)$ where Y is connected and locally path connected. Then $\exists\ f' : (Y, y_0) \to (E, e_0)$ lifting $f \Leftrightarrow f_\# \pi_1(Y, y_0) \subset p_\# \pi_1(E, e_0)$.*



Remark: X connected $\Rightarrow$ at most one such lift exists, by the Unique Lifting Theorem.
Proof: ($\Rightarrow$) Suppose f' exists. Then $f_\# = (pf')_\# = p_\# f'_\#$. Hence Im $f_\# \subset$ Im $p_\#$.

($\Leftarrow$) Suppose Im $f_\# \subset$ Im $p_\#$. For $y \in Y$ choose a path σ joining y_0 to y. Then $f \circ \sigma : I \to X$ joins x_0 to $f(y)$. Lift to a path $(f\sigma)'$ in E beginning at e_0 and define $f'(y) = (f\sigma)'(1)$.
Claim this gives a well-defined function of y:

Suppose $\tau : I \to Y$ also joins y_0 to y. Then $\sigma \cdot \tau^{-1}$ represents an element of $\pi_1(Y, y_0)$ so by hypothesis $\exists [w] \in \pi_1(E, e_0)$ s.t. $[p \circ \omega] = p_\#([w]) = f_\#([\sigma \cdot \tau^{-1}]) = [f \circ (\sigma \cdot \tau^{-1})]$. Since $p \circ w \simeq f \circ (\sigma \cdot \tau^{-1})$, lifting these paths to E beginning at e_0 results in paths with the same endpoint.

But w lifts $p \circ w$ and it ends at e_0 (it is a closed loop since it represents an element of $\pi_1(E, e_0)$). Hence the lift $\alpha : I \to E$ of $f \circ (\sigma \cdot \tau^{-1})$ beginning at e_0 also ends at e_0. Let $e_1 = \alpha(1/2)$.

The restriction of α to $[0, 1/2]$ lifts σ (beginning at e_0, ending at e_1).

The restriction of α to $[1/2, 1]$ lifts τ^{-1} (beginning at e_1, ending at e_0).

So the curve lifting τ beginning at e_0 ends at e_1. So using either σ or τ in the definition of $f'(y)$ results in $f'(y) = e_1$. Hence f' is well defined. $\surd$

To help show f' continuous:

Lemma 8.2.5 *Let $y, z \in Y$ and let γ be a path in Y from y to z. If the path $f \circ \gamma$ is contained in some evenly covered set U of X then $f'(y), f'(z)$ lie in the same sheet in $p^{-1}(U)$.*

Proof: Let $(f \circ \gamma)'$ be the lift of $f \circ \gamma$ beginning at $f'(y)$.
Claim: $(f \circ \gamma)'$ ends at $f'(z)$.

Proof of Claim: Use $\sigma \circ \gamma$ as the path joining y_0 to z in the definition of $f'(z)$. Then $(f \circ \sigma)' \cdot (f \circ \gamma)'$ is the lift of $f \circ (\gamma \cdot \sigma)$ which begins at e_0, so $f'(z)$ is the endpoint of $(f \circ \sigma)' \cdot (f \circ \gamma)'$, in other words the endpoint of $(f \circ \gamma)'$. $\surd$

Let S be the sheet of $p^{-1}(U)$ containing $f'(y)$.

$p|_S$ is a homeomorphism, which implies S contains the entire path $(f \circ \gamma)'$, so in particular it contains $f'(z)$. $\square$

Claim: f' is continuous.

Given $e \in E$, let $U_{p(e)} \subset X$ be an evenly covered set containing $p(e)$ and let S_e be the sheet in $p^{-1}(U_{p(e)})$ which contains e.

For an open set $V \subset E$, $V = \bigcup_{e \in V}(S_e \cap V)$, so to show f' is continuous, it suffices to show $f'^{-1}(W)$ is open whenever $W \subset E$ is open in some S_e.

Since $p|_{S_e}$ is a homeomorphism, $p(W)$ is open in X and is evenly covered (being a subset of the evenly covered set $U_{p(e)}$).

Set $A := f^{-1}\big(p(W)\big) \subset Y$. By continuity of f, A is open so its path components are open by hypothesis.

$(f')^{-1}(W) \subset A$. Show $(f')^{-1}(W)$ is open by showing $(f')^{-1}(W)$ is a union of path components of A.

Write $A = \bigcup_{i \in I} A_i$ where A_i is a path component of A.

Claim: $\forall i$, either $A_i \cap (f')^{-1}(W) = \emptyset$ or $A_i \subset (f')^{-1}(W)$.

Note: This shows $(f')^{-1}(W)$ is the union of those A_i which intersect it, thus completing the proof.

Proof of Claim: Suppose $y \in A_i \cap (f')^{-1}(W)$. Let $z \in A_i$. Show $z \in (f')^{-1}(W)$.

Let γ be a path joining y to z in A_i. (A_i is a path component so is path connected.)

Since $A_i \subset A = f^{-1}\big(p(W)\big)$, $f \circ \gamma$ is entirely contained in the evenly covered set $p(W)$, so by the Lemma, $f'(y)$ and $f'(z)$ lie in the same sheet of $p^{-1}\big(p(W)\big)$.

$y \in (f')^{-1}(W) \Rightarrow$ that sheet is W so $z \in (f')^{-1}(W)$. $\square$

Lemma 8.2.6 *A covering space of a locally path connected space is locally path connected.*

Proof: Let $E \xrightarrow{p} X$ be a covering projection, with X locally path connected.

Let V be open in E, let A be a path component of V and let $a \in A$.

Let $U \subset X$ be an evenly covered set containing $p(A)$ and let S be the sheet in $p^{-1}(U)$ containing a.

Replacing U by the smaller evenly covered set $p(S \cap V)$, we may assume $S \subset V$.

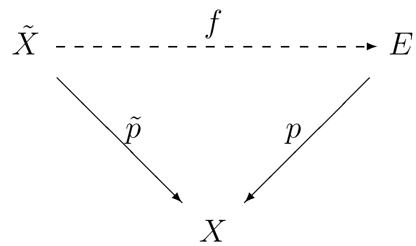
Let W be the path component of U containing $p(a)$. Hence W is open by hypothesis. $p|_S$ is a homeomorphism, so $B := p^{-1}(W) \cap S$ is a path connected open subset in E.

B is path connected, and $a \in B$, so $B \subset A$. Since B is open, $a \in \mathring{A}$ so A is open. $\square$

Corollary 8.2.7 *(***of Lifting Theorem***): A simply connected locally path connected covering space is a universal covering space.*

Proof: Let $\tilde{p} : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering projection s.t. $\tilde{X}$ is simply connected and locally path connected. Let $p : (E, e_0) \to (X, x_0)$ be a covering projection of X.

$\pi_1(\tilde{X}, \tilde{x_0}) = 1$ so the hypothesis $\tilde{p}_\# \pi_1(\tilde{X}, \tilde{x_0}) \subset p_\# \pi_1(E, e_0)$ of the Lifting Theorem is trivial. Hence $\exists f : \tilde{X} \to E$ s.t.



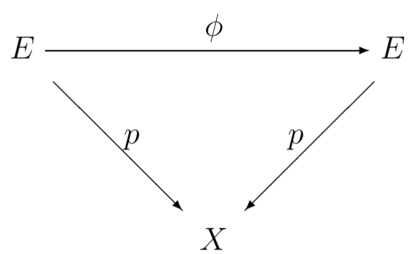
The Unique Lifting Theorem shows f is unique. □

Corollary 8.2.8 *(***of Lifting Theorem***:) Let W be simply connected and let $(E, e_0) \xrightarrow{p} (X, x_0)$ be a covering projection. Then $[(W, w_0), (E, e_0)] \xrightarrow{p_\#} [(W, w_0), (X, x_0)]$ is a set bijection.*

Proof: Essentially the same as the proof of Corollary 8.2.7. □

8.2.1 Computing Fundamental Groups from Covering Spaces

Definition 8.2.9 *Let $p : E \to X$ be a covering projection. A self-homeomorphism $\phi : E \to E$ is called a* covering transformation *if*



commutes.

Remark: $p\phi = p$ guarantees that $\forall x \in X$, ϕ is a self-map of $p^{-1}(x)$. $p^{-1}(x)$ is often called the *fibre* over x.

{ covering transformations of $E \xrightarrow{p} X$ } forms a group under composition.

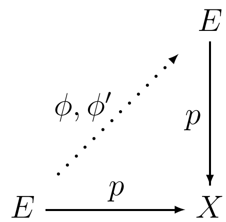
Example 1: $\exp : \mathbb{R} \to S^1$. The group of covering transformations is $\mathbb{Z}$.

Example 2: $p : S^n \to \mathbb{R}P^n$. The group of covering transformations is $\mathbb{Z}_2$, because it is the collection of maps sending $x \to x$ or $x \to -x$ (for $x \in S^n$).

Notice that in each case $|G| = \text{card}\left(p^{-1}(x)\right)$.

Lemma 8.2.10 *Let $p : E \to X$ be a covering projection with E connected. Let $\phi, \phi' : E \to E$ s.t. $p\phi = p$, $p\phi' = p$. If $\phi(e) = \phi'(e)$ for some $e \in E$ then $\phi = \phi'$. In particular, a covering transformation is determined by its value at any point.*

Proof:



Apply the Unique Lifting Theorem with $y_0 = e$ and $x_0 = \phi(e) = \phi'(e)$.

□

Theorem 8.2.11 *Let $p : E \to X$ be a covering projection s.t. E is simply connected and locally path connected (thus a universal covering space). Then $\pi_1(X)$ = group of covering transformations of p.*

(Since "simply connected" includes "path connected", notice that p onto implies that X is path connected, so $\pi_1(X)$ is well defined, i.e. independent of the choice of basepoint.)

Proof: Let G be the group of covering tranformations of p. Define $\psi : G \to \pi_1(X)$ as follows: Given $\phi \in G$, select a path w_ϕ joining e_0 to $\phi(e_0)$.

$p\phi(e_0) = pe_0 = x_0 \Rightarrow p \circ w_\phi$ is a closed loop in X so it represents an element of $\pi_1(X, x_0)$.

Define $\psi(\phi) = [p \circ w_\phi]$.

Claim: ψ is well-defined.

Proof: (of Claim:) If w'_ϕ is another path joining e_0 to $\phi(e_0)$ then E is simply connected $\Rightarrow w_\phi \simeq w'_\phi \,\text{rel}\{0, 1\}$.

Hence $p \circ w_\phi \simeq p \circ w_{\phi'} \,\text{rel}\{0, 1\}$. i.e. $[p \circ \omega_\phi] = [p \circ \omega'_\phi]$ in $\pi_1(X)$.

Claim: ψ is a group homomorphism.

Proof: (of Claim:) Let $\phi_1, \phi_2 \in G$. Pick paths w_{ϕ_1}, w_{ϕ_2} as above joining e_0 to $\phi_1(e_0)$ resp. , $\phi_2(e_0)$. Then $\phi_1 \circ w_{\phi_2}$ is a path joining $\phi_1(e_0)$ to $\phi_1\big(\phi_2(e_0)\big) = \phi_1\phi_2(e_0)$. So we use $w_{\phi_1} \dot{(} \phi_1 \circ w_{\phi_2})$ to define $\psi(\phi_1\phi_2)$.

ϕ is a covering transformation, so $p \circ \phi_1 \circ w_{\phi_2} = p \circ w_{\phi_2}$.

Hence $\psi(\phi_1\phi_2) = [p \circ (w_{\phi_1} \cdot (\phi_1 \circ w_{\phi_2})] = [p \circ w_{\phi_1}][p \circ \phi_1 \circ w_{\phi_2}]$

$= [p \circ w_{\phi_1}][p \circ w_{\phi_2}]$

$= \psi(\phi_1)\psi(\phi_2)$.

Claim: ψ is injective.

Proof: (of Claim:) $\psi(\phi_1) = \psi(\phi_2) \Rightarrow p \circ w_{\phi_1} \simeq p \circ w_{\phi_2}$. This implies the lifts of w_{ϕ_1} and w_{ϕ_2} beginning at e_0 must end at the same point.

Hence $\phi_1(e_0) = \phi_2(e_0)$ which implies $\phi_1 = \phi_2$.

Claim: ψ is surjective.

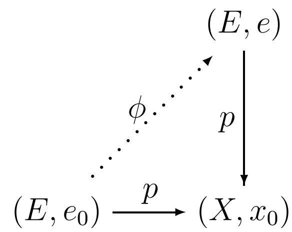
Proof: (of Claim:) Let $[\sigma] \in \pi_1(X, x_0)$.

Lift σ to a path σ' in E beginning at e_0.

Let $e = \sigma'(1)$.

It suffices to show there exists a covering transformation $\phi : E \to E$ s.t. $\phi(e_0) = e$.

Then we use σ' to define $\psi(\phi)$ to see that $\psi(\phi) = \sigma$.



Since E is connected and locally path connected and $1 = p_\# \pi_1(E, e_0) \subset p_\# \pi_1(E, e)$, the lifting theorem implies $\exists \phi$ s.t. $p \circ \phi = p$ and $\phi(e_0) = e$.

It remains to show ϕ is a homeomorphism.

But we may apply the lifting theorem again with the roles of e_0 and e reversed to get $\theta : (E, e) \to (E, e_0)$.

Then $p \circ \theta \circ \phi = p$ and $\theta \circ \phi(e_0) = e_0$ so by the previous Lemma, $\theta \circ \phi = 1_E$. Similarly $\phi \circ \theta = 1_E$. So ϕ is a homeomorphism. □

Remark: We already used this to show that $\pi_1(S^1) = \mathbb{Z}$. Later we will show that S^n is simply connected for $n \geq 2$, so that the theorem applies to $S^n \to \mathbb{R}P^n$, giving $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for $n \geq 2$.

Note: The preceding proof showed a bijection between covering transformations and elements of $p^{-1}(x_0)$. Each point corresponds to a covering transformation taking e_0 to that point.

8.2.2 'Galois' Theory of Covering Spaces

Theorem 8.2.12 *Let $p : E \to X$ be a covering projection s.t. E is simply connected and locally path connected (thus a universal covering space). Then for every subgroup $H \subset \pi_1(X)$, $\exists$ a covering projection $p_H : E_H \to X$, unique up to equivalence of covering spaces, such that $(p_H)_\#(\pi_1(E_H)) = H$.*

Proof: $\{$covering transformations of $E\} \cong \pi_1(X)$ so H can be regarded as the set of covering transformations of E. Hence H acts on E. Let $E_H = E/H$.

If $e' = h \circ e$ for $h \in H$, since h is a covering transformation, $p(e') = p(e)$.

Hence p induces a well defined map $p_H : E/H \to X$.

For evenly covered U_x of $p : E \to X$, sheets $p^{-1}(U_x)$ correspond bijectively to elements of $\pi_1(X)$.

$p_H^{-1}(U_x)$ is what we get by identifying S, S' whenever S, S' correspond to group elements g, g' s.t. $g' = gh$ for some $h \in H$ (in other words g' and g are in the same coset of G (mod H)).

Hence p_H is a covering projection (with U_x as evenly covered set).

Also Theorem 8.1.2 implies $E \xrightarrow{f} E/H$ is a covering projection. To apply the theorem we need to know that $\forall e \in E$, $\exists V_e$ s.t. $V_e \cap hV_e = \emptyset$ unless $h = 1$. Set $V_e :=$ the sheet over $U_{p(e)}$ which contains e for some evenly covered $U_{p(e)} \subset X$. This works since h is a covering translation so hS is also a sheet and sheets are disjoint.

By inspection, the group of covering translations of $f_H \cong H \cong \pi_1(E/H)$. (In general, the group of covering translations of $Y \to Y/G$ is isomorphic to G.)

By Corollary 8.1.10, any covering projection induces a monomorphism on π_1.

Hence $(p_H)_\# : H = \pi_1(E_H) \hookrightarrow \pi_1(E)$.

In other words $(p_H)_\#\big(\pi_1(E_H)\big) = H$. $\square$

8.2.3 Existence of Universal Covering Spaces

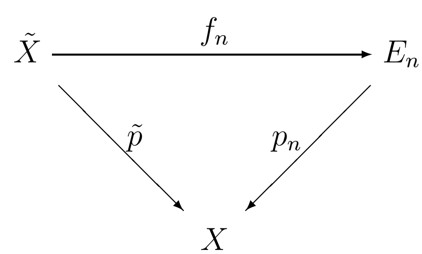
Not every space has a universal covering space.
Example: Let $X = \prod_{j=1}^{\infty} S^1$.
Proof: Let $E_n = \prod_{j=1}^{n} \mathbb{R} \times \prod_{j=n+1}^{\infty} S^1$.

It's easy to check that $p_n = \exp \times \cdots \times \exp \times 1\big|_{\prod_{j=n+1}^{\infty} S^1}$ is a covering projection.

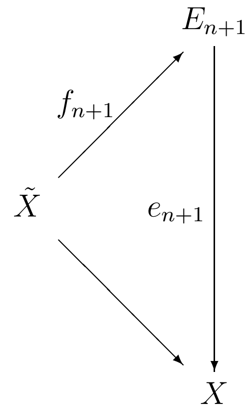
(In general a product of covering projections is a covering projection.)

Suppose X had a universal covering projection $\tilde{p} : \tilde{X} \to X$.

Then $\forall n$, we have



By uniqueness of f_n,



where e_{n+1} is exp on factor $(n+1)$ and the identity on the other factors.

Apply π_1 and use that $p_\#$ is a monomorphism to see that all maps on π_1 are monomorphisms.

$$\pi_1(\tilde{X}) \subset \cdots \subset \pi_1(E_{n+1}) \subset \pi_1(E_n) \subset \cdots \subset \pi_1(X).$$

$$\pi_1(X) = \prod_{j=1}^{\infty} \pi_1(S^1) = \prod_{j=1}^{\infty} \mathbb{Z}$$

and $\pi_1(E_n)$ is the subgroup $\prod_{j=n+1}^{\infty} \mathbb{Z}$. Hence $\pi_1(X) \subset \bigcap_{n=1}^{\infty} \pi_1(E_n) = 0$. So $\pi_1(X) = 0$.

Let $U \subset X$ be an evenly covered set for the covering projection $\tilde{X} \to X$.

Replace U by the basic open subset $U_1 \times U_2 \times \cdots \times U_n \times S^1 \times S^1 \times \ldots$

For $j = 1, \ldots, n$ select $u_j \in U_j$.

Define $\alpha : S^1 \to X$ by

$$\begin{cases} \alpha_j = c_{u_j} & j = 1, \ldots, n \\ \alpha_{n+1} = 1_{S^1} & \\ \alpha_j = c_* & j > n+1 \end{cases}$$

Notice that Im $(\alpha) \subset U$. $[\alpha] = (0, \ldots, 0, 1, 0, \ldots) \in \pi_1(X) = \prod_{j=1}^{\infty} \mathbb{Z}$ (where the '1' is in position $n+1$).

Let T be a sheet in $\tilde{p}^{-1}(U)$.

Im $(\alpha) \subset U$, $\tilde{p}|_T$ is a homeomorphism, so α has a lift α' which is a closed curve in T.

So α' represents a class in $\pi_1(\tilde{X})$ and $\tilde{p}_\#([\alpha']) = [\alpha]$. But $\pi_1(\tilde{X}) = 0$. This is a contradiction since $[\alpha] = (0, \ldots, 0, 1, 0, \ldots) \neq 0$.

Hence X has no universal covering space.

□

Definition 8.2.13 *A space X is called* semilocally simply connected *if each point $x \in X$ has an open neighbourhood U_x s.t. $i_\# : \pi_1(U_x, x) \to \pi_1(X, x)$ is the trivial map of groups. (where $i : U_x \hookrightarrow X$ denotes the inclusion).*

Notice that $\prod_{n=1}^{\infty} S^1$ is not semilocally simply connected.

Theorem 8.2.14 *Let X be connected, locally path connected and semilocally simply connected. Then X has a universal convering space.*

Proof: Choose $x_0 \in X$.

For path α, β in X beginning at x_0, define equiv. reln.: $\alpha \sim B$ if $\alpha(1) = \beta(1)$ and $\alpha \simeq \beta$ rel $(0, 1)$.

Let $\tilde{X} = \{\text{equiv. classes}\} \leftarrow$ (*paths beginning at* x_0)

Define $\tilde{p} : \tilde{X} \to X$.

$[\alpha] \to \alpha(1)$.

Topologize $\tilde{X}$ as follow: Given $[\alpha] \in \tilde{X}$ and open $V \subset X$ containing $\alpha(1)$, define subset denoted $\langle \alpha, V\rangle$ of $\tilde{X}$ by $\langle \alpha, V\rangle = \{[w] \in \tilde{X} | [w] = [\alpha \cdot \beta]$ for some path β in $V\}$. $\leftarrow$ *(strictly speaking mean* $\text{Im}\beta \subset V$.*)*

Note: $\langle \alpha, V\rangle$ is independent of choice of representation for $[\alpha]$ used to define it.

Claim: $\{\langle \alpha, V\rangle\}$ form a base for a topology on $\tilde{X}$.

Proof: Show intersection of 2 such sets is $\emptyset$ or a union of sets of this form.

Suppose $[w] \in \langle \alpha, V\rangle \cap \langle \alpha', V'\rangle \neq \emptyset$

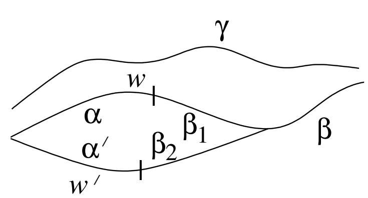
Suff. to show:

Claim: $\langle w, V \cap V'\rangle \subset \langle \alpha, V\rangle \cap \langle \alpha', V'\rangle$

Proof: Suppose $\gamma \in \langle w, V \cap V'\rangle$ $\quad \therefore \; [\gamma] = [w \cdot \beta]$ some β in $V \in V'$.

$[w] \in \langle \alpha, V\rangle \Rightarrow \exists \beta_1$ in V s.t. $[w] = [\alpha \cdot \beta_1]$

$[w] \in \langle \alpha', V'\rangle \Rightarrow \exists \beta_2$ in V' s.t. $[w] = [\alpha', \beta_2]$



where $w' \equiv \alpha' \cdot \beta_2 \simeq w$.

$\beta_1 \cdot \beta$ in V, $[\alpha] = [\alpha \cdot \beta_1 \cdot \beta_2] \Rightarrow [\gamma] \in \langle \alpha, V\rangle$.

Similarly $[\gamma] \in \langle \alpha', V'\rangle$. $\therefore \langle w, V \cap V'\rangle \subset \langle \alpha, V\rangle \cap \langle \alpha', V'\rangle$

Give $\tilde{X}$ the topology defined by this base.

Let $V \subset X$ be open.

Then $\tilde{p}^{-1}(V) = \{[w] \in \tilde{X} | w(1) \in V\} = \bigcup_{\{\alpha | \alpha(1) \in V\}} \langle \alpha, V \rangle$

$\therefore \tilde{p}$ cont.

For $x \in X$ find V_x s.t. $i_\# : \pi_1(V_x, x) \to \pi_1(X, x)$ is trivial. $i : V_x \mapsto X$

Let U_x = path component of V_x containing x. open since X locally path connected.

(A): Show $\tilde{p}^{-1}(U_x) = \coprod_{\{[\alpha] | \alpha(1) = x\}} \langle \alpha, U_x \rangle$.

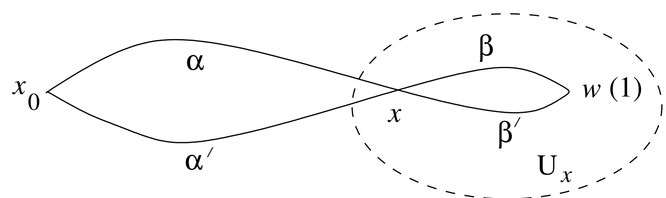
1. $\supset$ $\alpha(1) = x$. $\tilde{p}([w]) = w(1) \in U_x$.
2. $\subset$ Suppose $[w] \in \tilde{X}$ s.t. $\tilde{p}[w] \in U_x$. *i.e.* $[w] \in p^{-1}(U_x)$

   Then $\exists$ path β in U_x joining x to $w(1)$.

   Let $\alpha = w \cdot \beta^{-1}$. $[\alpha \cdot \beta] = [w \cdot \beta^{-1} \cdot \beta] = [w]$.

   $\therefore [w] \in \langle \alpha, U_x \rangle \subset \bigcup_{\alpha(1) = x} \langle \alpha, U_x \rangle \leftarrow$ (*α ends where β begins – at x)*
3. union is disjoint Suppose $[w] \in \langle \alpha, U_x \rangle \cap \langle \alpha', U_x \rangle$

   $[\alpha' \cdot \beta'] = [w] = [\alpha \cdot \beta]$ $\quad \beta$, β' paths in U_x



$U_x \subset V_x \Rightarrow$ path $\beta \cdot \beta'^{-1}$ reps. elt. of $\pi_1(V_x, x)$ so choice of $V_x \Rightarrow [\beta \cdot \beta'^{-1}] = [c_x]$ in $\pi_1(X, x)$.

$\therefore$ $[\alpha] = [\alpha \cdot \beta \cdot \beta'^{-1}] = [w \cdot \beta'^{-1}] = [\alpha' \cdot \beta' \cdot \beta'^{-1}] = [\alpha']$.

(B) Show $\forall$ $[\alpha]$ s.t. $\alpha(1) = x$ that $\tilde{p}|_{\langle \alpha, U_x \rangle} : \langle \alpha, U_x \rangle \to U_x$ is a homeomorphism.

Any pt. in U_x can be joined to x by a path in U_x, hence q is onto.

Claim: q is $1-1$.

Suppose $[w]$, $[w'] \in \langle \alpha, U_x \rangle$ s.t. $q([w]) = q([w'])$.

Find paths β, β' in U_x s.t. $[w] = [\alpha \cdot \beta]$, $[w'] = [\alpha \cdot \beta']$.

β, β' each join x to $w(1) = w'(1)$ in U_x so as above $[\beta^{-1} \cdot \beta'] = [c_x]$ in $\pi_1(X, x)$.

$\therefore [w] = [\alpha \cdot \beta] = [\alpha \cdot \beta \cdot \beta^{-1} \cdot \beta'] = [\alpha \cdot \beta'] = [w']$.

Claim: q^{-1} is continuous.

Let $\langle\gamma, V\rangle$ be basic open set with $\langle\gamma, V\rangle \subset \langle\alpha, U_x\rangle$.

$q(\langle\gamma, V\rangle) =$ path component of x within $V \cap U_x$ open since X locally path connected.

Note: $q\langle\gamma, V\rangle = \langle\gamma$, path component of $\gamma(1)$ within $V\rangle$. This implies we may assume V is path connected.

$q(w) = \beta(1)$ where β in V, $\beta(1) \in U_x$, and $\beta(0) = \alpha(1) = x$ since $\beta \in \langle\gamma, V\rangle \subset \langle\alpha, U_x\rangle$.

$\Rightarrow\ q(\langle\gamma, V\rangle) \subset V \cap U_x$.

Conversely $V \cap U_x \subset q(\langle\gamma, V\rangle)$ since endpt. of γ can be joined to $\beta(1)$ by path in V.

$\therefore q^{-1}$ cont.

$\therefore \tilde{p} : \tilde{X} \to X$ covering proj.

$\therefore$ Suff. to show:

(C) $\tilde{X}$ is simply connected:

Pick $\tilde{x}_0 := [c_{x_0}] \in \tilde{X}$ as basept. of $\tilde{X}$.

1. $\tilde{X}$ is path connected:

   Given $[w] \in \tilde{X}$, define $I \xrightarrow{\phi_w} \tilde{X}$ by $\phi_w(s) = [w_s]$ where $w_s(t) = w(st)$.
   $w_0 = c_{x_0}$, $w_1 = w$.

   $$\therefore\ \phi_w(0) = [w_0] = [c_{x_0}] = \tilde{x}_0\ .$$

   Hence

   $$\phi_w \text{ joins } \tilde{x}_0 \text{ to } [w].$$

   $$\therefore\ \phi_w(1) = [w_1] = [w].$$

   $\therefore\ \tilde{X}$ path connected.

   Before showing $\pi_1(\tilde{X}, \tilde{x}_0) = 1$ need properties of ϕ_w.

   (a) $\tilde{p} \circ \phi_w(s) = \tilde{p}([w_s]) = w_s[1] = w(s) \Rightarrow \phi_w$ is the lift of w to $\tilde{X}$ beginning at $\tilde{x}_0$.

   (b) Claim: $[w] = [\gamma] \Rightarrow \emptyset_w \simeq \emptyset_\gamma$ rel $(0, 1)$.
   Proof: Follows from Covering Homotopy Thm.

2. Show $\pi_1(\tilde{X}, \tilde{x}_0) = 1$. Let σ rep. an elt. of $\pi_1(\tilde{X}, \tilde{x}_0)$. Then $\tilde{p} \circ \sigma$ is a path in X joining x_0 to itself. $\therefore \sigma$, $\phi_{\tilde{p}\circ\sigma}$ are both lifts of $\tilde{p} \circ \sigma$ to $\tilde{X}$ beginning at $\tilde{x}_0$. $\therefore$ Unique lifting $\Rightarrow\ \sigma = \phi_{\tilde{p}\circ\sigma} \Rightarrow \sigma(1) = \phi_{\tilde{p}\circ\sigma}(1)$ and $\tilde{x}_0 = \sigma(1)$ because σ represents an element of $\pi_1(X, \tilde{x}_0)$. $\pi_1(X, \tilde{x}_0)$.)

Therefore in $\tilde{X}$, $[\tilde{p} \circ \sigma] = [(\tilde{p} \circ \sigma)_1] = \phi_{\tilde{p}\circ\sigma}(1) = \tilde{x}_0 = [c_{x_0}]$.

Therefore $\sigma = \phi_{\tilde{p}\circ\sigma} \overset{\text{part (b) above}}{\simeq} \phi_{c_{x_0}} = c_{\tilde{x}_0}$ so $[\sigma] = 1$ in $\pi_1(\tilde{X}, \tilde{x}_0)$.

Therefore $\tilde{X}$ is simply connected. $\surd$

(So by Corollary 8.2.7, being a simple connected cover of a connected, path connected and locally path connected space, $\tilde{X}$ is a universal covering space.) $\square$