# Chapter 6

# Connectedness

**Definition 6.0.40** *A pair of nonempty open subsets $A$ and $B$ of a topological space $X$ is called a* disconnection *of $X$ if $A \cap B = \emptyset$ and $A \cup B = X$.*

**Note:** If $A, B$ is a disconnection of $X$ then $A$ and $B$ are also closed since $A = B^c$ and $B = A^c$.

**Proposition 6.0.41** *A subspace of $\mathbb{R}$ is connected $\Leftrightarrow$ it is an interval. In particular $\mathbb{R}$ is connected.*

**Proof:** Exercise.

**Proposition 6.0.42** *Suppose $f : X \to Y$ is continuous. If $X$ is connected then $f(X)$ is connected.*

**Proof:** Exercise.

**Proposition 6.0.43** *Suppose $f : X \to Y$ is continuous. If $X$ is connected then $f(X)$ is connected.*

**Proof:** Assume there is a disconnection $G, H$ of $f(X)$. Then $f^{-1}(G)$, $f^{-1}(H)$ is a disconnection of $f(X)$. This is a contradiction, so $f(X)$ must be connected. □

**Proposition 6.0.44** *Suppose $A \subset X$. If $A$ is connected then $\bar{A}$ is also connected.*

**Proof:** Suppose $G, H$ is a disconnection of $\bar{A}$. Then $G \cap A, H \cap A$ is a disconnection of $A$. (Note that $G \cap \bar{A} \neq \emptyset \Rightarrow G \cap A \neq \emptyset$. Similarly for $H$.) □

**Proposition 6.0.45** *If $X_\alpha$ is connected $\forall \alpha$, and $\cap_\alpha X_\alpha \neq \emptyset$, then $\cup_\alpha X_\alpha$ is connected.*

**Proof:** Suppose $G, H$ is a disconnection of $\cup_\alpha X_\alpha$. Then $\forall \alpha$, $X_\alpha = (G \cap X_\alpha) \cup (H \cap X_\alpha)$. Hence either $G \cap X_\alpha = \emptyset$ or $H \cap X_\alpha = \emptyset$. If $H \cap X_\alpha = \emptyset$, then $X_\alpha = G \cap X_\alpha$ so $X_\alpha \subset G$. Otherwise $X_\alpha \subset H$. In other words, each $X_\alpha$ is in one of the sets $G, H$. Since $\cap_\alpha X_\alpha \neq \emptyset$ and $G \cap H = \emptyset$, each $X_\alpha$ is in the same set, say $G$. But then $\cup_\alpha X_\alpha \subset G$ so that $H = G^c = \emptyset$, which is a contradiction. Hence $\cup_\alpha X_\alpha$ is connected. □

**Lemma 6.0.46** *Let $X$ be disconnected. Then $\exists f : X \to \{0,1\}$ which is onto.*

**Proof:** Let $A, B$ be a disconnection. Define $f(x) = 0, x \in A$ and $f(x) = 1, x \in B$. $\square$

**Theorem 6.0.47** *Let $X$ $\prod_{\alpha\in J} X_\alpha$. Then $X$ is connected $\Leftrightarrow$ $X_\alpha$ is connected $\forall\alpha$.*

**Proof:** ($\Longrightarrow$) Suppose $X$ is connected. Then $X_\alpha = \pi_\alpha(X)$ is connected.

($\Longleftarrow$) Suppose $X_\alpha$ is connected $\forall\alpha$. Assume $X$ is disconnected. Let $f : X \to \{0,1\}$ be onto. Pick $x_\alpha \in X_\alpha$. (The theorem is trivial if $X_\alpha = \emptyset$ for some $\alpha$.)

For $\alpha \in J$ and $x \in X$, define $\imath_{\alpha_0} : X_{\alpha_0} \to X$ by

$$\pi_\alpha(\imath_{\alpha_0}(w)) = w \text{ for } \alpha = \alpha_0$$

and

$$\pi_\alpha(\imath_{\alpha_0}(w)) = x_\alpha \text{ for } \alpha \neq \alpha_0.$$

Then

$$X_{\alpha_0} \xrightarrow{\imath_{\alpha_0}} X \xrightarrow{f} \{0,1\}$$

is continuous, so $X_{\alpha_0}$ is connected $\Rightarrow f\imath_{\alpha_0}(X_{\alpha_0})$ is connected.

Then $f\imath_{\alpha_0}$ must not be onto since $\{0,1\}$ is disconnected.

Therefore $\forall w \in X_{\alpha_0}$, $f \circ \imath_{\alpha_0}(w) = f \circ \imath_{\alpha_0}(x_{\alpha_0}) = f(x)$.

In other words, if $x, y \in X$ and $x_\alpha = y_\alpha$ for $\alpha \neq \alpha_0$ then $f(x) = f(y)$.

This is true $\forall\alpha_0$ so $f(x) = f(y)$ whenever $x$ and $y$ differ in only one coordinate.

By induction, $f(x) = f(y)$ whenever $x, y$ differ in only finitely many coordinates.

**Claim:** Given $z \in X$, $\{y \in X | y_\alpha = z_\alpha$ for almost all $\alpha\}$ is dense in $X$.

**Proof (of Claim):** Every open set $V$ contains a basic open set $U = \prod_\alpha U_\alpha$ with $U_\alpha = X_\alpha$ for almost all $\alpha$. Hence $\exists y \in U^c$ s.t. $y_\alpha = z_\alpha$ for almost all $\alpha$. $\surd$

Since $\{0,1\}$ is Hausdorff, $f(y) = f(z)$ $\forall y$ in a dense subset $\Rightarrow f(y) = f(z)$ $\forall y \in X$. Hence $f$ is constant. Since $f$ is onto, this is a contradiction. So $X$ is connected. $\square$

### 6.0.2 Components

**Definition 6.0.48** *A (connected) component of a space $X$ is a maximal connected subspace.*

**Theorem 6.0.49**

*1. Each nonempty connected subset of $X$ is contained in exactly one component. In particular each point of $X$ is in a unique component so $X$ is the union of its components.*

*2. Each component of $X$ is closed.*

*3. Any nonempty connected subspace of $X$ which is both open and closed is a component.*

**Proof:**

1. Let $\emptyset \neq Y \subset X$ be connected. Let $C = \bigcup_{A \text{ connected}, Y \subset A} A$.

Since $Y \subset \bigcap_{A \text{ connected}, Y \subset A} A$, this intersection is non-empty, so by the earlier Proposition, $C$ is connected. $C$ is a component containing $Y$. If $C'$ is another component containing $Y$ then by construction $C' \subset C$ so $C' = C$ by maximality.

2. If $C$ is a component then $\bar{C}$ is connected by the earlier Proposition, and $C \subset \bar{C}$ so $C = \bar{C}$ by maximality. Hence $C$ is closed.

3. Suppose $\emptyset \neq Y$ with $Y$ connected, and both closed and open. Let $C$ be the component of $X$ containing $Y$. Let $A = C \cap Y$ and $B = C \cap Y^c$. Since $Y$ and $Y^c$ are open, we must have $C \cap Y^c = \emptyset$ so that $A, B$ is not a disconnection of $C$. Hence $C = C \cap Y$ so $C \subset Y$. So $Y = C$ is a component.

□

Note: A component need not be open. For example, in $\mathbb{Q}$ the components are single points.

### 6.0.3 Path Connectedness

Notation: Let $I = [0, 1]$.

**Definition 6.0.50** *$X$ is called* path connected *if $\forall x, y \in X$ $\exists w : I \to X$ s.t. $w(0) = x$, $w(1) = y$.*

**Proposition 6.0.51** *Path connected $\Rightarrow$ connected.*

**Proof:** Suppose $X$ is path connected. If $X$ is not connected, then $X$ has at least two components $C_1, C_2$. Pick $x \in C_1$, $y \in C_2$ and find $w : I \to X$ s.t. $w(0) = x$, $w(1) = y$. $I$ is connected, so $w(I)$ is connected, so by an earlier Proposition, $w(I)$ is contained in a single component. This is a contradiction, so $X$ is connected. □

**Example:** A connected space need not be path connected.

Let $Y = \{(0, y) \in \mathbb{R}^2\}$ (the $y$-axis)
$Z = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$ the graph of $y = \sin(1/x)$ on $(0, 1]$
$X = Y \cup Z$.

(a) $X$ is connected:

**Proof:** The map $(0, 1] \xrightarrow{f} \mathbb{R}^2$ given by $t \mapsto (t, \sin(1/t))$ is continuous so $Z = \mathrm{Im}(f)$ is connected.

Hence $\overline{Z}$ is connected.

$0 \in \overline{Z}$. But $0 \in Y$ so $Y \cap \bar{Z} \neq \emptyset$ and $Y$ is connected. Hence $Y \cap \overline{Z}$ is connected. But $Y \cap Z = Y \cap \overline{Z}$ since the limit points of $Z$ are in $Y$.

(b) $X$ is not path connected:

**Proof:** Suppose $w : I \to X$ s.t. $w(0) = (0, 0)$ and $w(1) = (1, \sin(1))$.

Let $t_0 = \inf\{t|w(t) \in Z\}$.

$t < t_0 \Rightarrow w(t) \in Y$ and $Y$ is closed so by continuity $w(t_0) \in Y$.

By definition of inf, $\forall \delta > 0 i\ \exists 0 < r > \delta$ s.t. $\omega(t_0 + r) = \big(a, \sin(1.a)\big) \in Z$ for some $a$. Then $\pi_x \omega[t_0, t_0 + r]$ contains 0 and $a$ and is connected so it contains all $x$ in $[0, a]$. In particular, $\omega[t_0, t_0 + \delta) \supset \omega[t_0, t_0 + r]$ contains points of the form $(*, 0)$ and points of the form $(*, 1)$. This is true for all $\delta$, so $w$ is not continuous at $t_0$. This is a contradiction, so $X$ is not path connected.

□

Note that from this example, $A \subset X$ is path connected does not always imply $\bar{A}$ is path connected. (Let $A = Z$ in the above example.)

**Proposition 6.0.52** *If $f : X \to Y$ is continuous and $X$ is connected, then $f(X)$ is path connected.*

**Proof:** Given $f(x_1), f(x_2) \in f(X)$ let $w$ be a path connecting $x_1$ and $x_2$. Then $f \circ w : I \to Y$ connects $f(x_1)$ and $f(x_2)$.

□

**Proposition 6.0.53**

1. *If $X_\alpha$ is path connected $\forall \alpha$, then $\cap_\alpha X_\alpha \neq \emptyset \Rightarrow \cup_\alpha X_\alpha$ is path connected.*
2. *$\prod_\alpha X_\alpha$ is path connected $\Leftrightarrow X_\alpha$ is path connected $\forall \alpha$.*

**Proof:**

1. Let $a \in \cap_\alpha X_\alpha$. Given $x, y \in \cup_\alpha X_\alpha$, connect them to each other by connecting each to $a$.

2. Let $X = \prod_\alpha X_\alpha$.

($\Longrightarrow$) Suppose $X$ is path connected. Then $X_\alpha = \pi_\alpha(X)$ is path connected.

($\Longleftarrow$) Suppose $X_\alpha$ is path connected $\forall \alpha$. Given $x = (x_\alpha), y = (y_\alpha) \in X$, $\forall \alpha$ select $w_\alpha : I \to X_\alpha$ s.t. $w_\alpha(0) = x_\alpha$, $w_\alpha(1) = y_\alpha$.

Define $w : I \to X$ by $\pi_\alpha \circ w = w_\alpha$. Then $w$ is continuous since each projection is, and $w(0) = x$ and $w(1) = y$.

□

**Definition 6.0.54** *A* path component *of a space $X$ is a maximal path connected space.*

**Proposition 6.0.55** *Each path connected subset of $X$ is contained in exactly one path component. In particular each point of $X$ is in a unique path component, so $X$ is the union of its path components.*

**Proof:** Insert "path" before "connected" and before "component" in the earlier proof, since it used only that $\cap_\alpha X_\alpha \neq \emptyset$ with $X_\alpha$ connected implies $\cup_\alpha X_\alpha$ connected.

□

## 6.1 Local Properties

**Definition 6.1.1** *A space $X$ is called* locally compact *if every point has a neighbourhood whose closure is compact.*

**Example:** $\mathbb{R}^n$ is locally compact, but not compact.

**Proposition 6.1.2** *If a space $X$ is compact, then it is locally compact.*

(The proof is obvious.)

**Theorem 6.1.3** *Let $X$ be a locally compact Hausdorff space. Then $\exists$ a compact Hausdorff space $X$ and an inclusion $\imath : X \longrightarrow X_\infty$ s.t. $X_\infty \setminus X$ is a single point.*

**Proof:** Let $\infty$ denote an element not in the set $X$ and define $X_\infty = X \cup \{\infty\}$ as a set. Topologize $X_\infty$ by declaring the following subsets to be open:

(i) $\{U \mid U \subset X$ and $U$ open in $X\}$

(ii) $\{V \mid V^c \subset X$ and $V^c$ is compact$\}$

(iii) the full space $X_\infty$

**Exercise:** Check this is a topology.

**Claim:** $X_\infty$ is compact.

**Proof:** Let $\{U_\alpha\}$ be an open cover of $X_\infty$. If some $U_\alpha$ is $X_\infty$ itself, it is a finite subcover so we are finished. Suppose not. Find $U_{\alpha_0}$ s.t. $\infty \in U_{\alpha_0}$. $U_{\alpha_0}$ must be a set of type (ii) so $U^c_{\alpha_0}$ is a compact subset of $X$.

$\{U_\alpha \cap X\}$ covers $U^c_{\alpha_0}$ so there is a finite subcover $\{U_{\alpha_1} \cap X, \ldots, U_{\alpha_n} \cap X\}$. But then $\{U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ covers $X_\infty$.

I claim that $X_\infty$ is Hausdorff.

**Proof:** Let $x \neq y \in X_\infty$. If $x, y \in X$, we can separate them using the open sets from $X$, so say $y = \infty$.

Since $X$ is locally compact, $\exists U$ s.t. $x \in U$ and $\bar{U}$ is a compact subset of $X$. Hence $X_\infty \setminus \bar{U}$ is open in $X_\infty$ and $\infty \in X_\infty \setminus \bar{U}$.

**Definition 6.1.4** *Given a locally compact Hausdorff space $X$, the space $X_\infty$ formed by the above construction is called the* one point compactification *of $X$.*

**Example:** If $X = \mathbb{R}^n$ then $X_\infty$ is homeomorphic to $S^n$. (The inverse homeomorphism is given by stereographic projection.)

**Corollary 6.1.5** *Suppose $X$ is locally compact and Hausdorff, and $A \subset X$ is compact. If $U$ is open s.t. $A \subset U$ and $U \neq X$, then $\exists f : X \to [0,1]$ s.t. $f(A) = 0$ and $f(U^c) = 1$.*

**Proof:** $X_\infty$ is normal so $\exists$ such an $f$ on $X_\infty$ by Urysohn. Restrict $f$ to $X$. □

**Definition 6.1.6** *A space $X$ is called* locally [path] connected *if the [path] components of open sets are open.*

**Proposition 6.1.7** *$X$ is locally [path] connected $\Leftrightarrow \forall x \in X$ and $\forall$ open $U$ containing $x$, $\exists$ a [path] connected open $V$ s.t. $x \in V \subset U$.*

**Proof:** ($\Longrightarrow$) Given $x \in U$, Let $V$ be the [path] component of $U$ containing $x$.

($\Longleftarrow$) Let $U$ be open. Let $C$ be a [path] component of $U$ and let $x \in C$. There exists an open [path] connected $V$ s.t. $x \in V \subset U$ so by maximality of [path] components, $V \subset C$.

Hence $x \in \overset{\circ}{C}$. This is true $\forall x \in C$ so $C$ is open. □

Note:

1. Locally [path] connected does not imply [path] connected.

For example, $[0,1] \cup [2,3]$ is locally [path] connected but not [path] connected.

2. Conversely [path] connected does not imply locally [path ] connected.

For example, the *comb space*

$$X = \{(1/n, y) \mid n \geq 1,\ 0 \leq y \leq 1\} \cup \{(0, y) \mid 0 \leq y \leq 1\} \cup \{(x, 0) \mid 0 \leq y \leq 1\}$$

$X$ is [path] connected but not locally [path] connected.

Another example is the union of the graph of $\sin(1/x)$ with the $y$-axis and a path from the $y$-axis to $(1, \sin(1))$. Without this path, the space is not path connected.

**Proposition 6.1.8** *If $X$ is locally path connected, then $X$ is locally connected.*

**Proof:** $\forall U$ and $\forall x \in U$ $\exists$ a path connected $V$ s.t. $x \in V \subset U$. But $V$ is connected since path connected implies connected. □

**Proposition 6.1.9** *If $X$ is connected and locally path connected, then $X$ is path connected.*

**Proof:** Let $C$ be a path component of $X$. Hence $C$ is open (by definition of locally path connected applied to the open set $X$).

Let $x \in \bar{C}$.

$X$ is locally path connected $\Rightarrow \exists$ a connected open set $U$ containing $x$. (Apply the definition of locally path connected to the open set $X$. The component of $X$ containing $x$ is open.)

$x \in \bar{C} \Rightarrow U \cap C \neq \emptyset \Rightarrow C \cup U$ is path connected.

So $C \cup U = C$ (by maximality of components)

Hence $x \in U \subset C$ and therefore $C = \bar{C}$, in other words $C$ is closed.

Since $C$ is both open and closed, by theorem 6.0.49, $C$ is a connected component.

Since $X$ is connected, $C = X$.

Hence $X$ is path connected.

□