Chapter 2

Compactness

Definition 2.0.7 *A topological space X is called compact if it has the property that every open cover of X has a finite subcover.*

Theorem 2.0.8 Heine-Borel *A subset $X \subset \mathbb{R}^n$ is closed and bounded if and only if every open cover of X has a finite cover.*

Proposition 2.0.9 *Given a basis for the topology on X, X is compact $\Leftrightarrow$ every open cover of X by sets from the basis has a finite subcover.*

$\Rightarrow$ Obvious

$\Leftarrow$ Let U_α be an open cover of X.

Write each U_α as a union of sets in the basis to get a cover of X by basic open sets.

Select a finite subcover $V_1, \ldots, V_n$ from these.

By construction $\forall j$ $\exists \alpha_j$ s.t. $U_{\alpha_1}, \ldots, U_{\alpha_n}$ cover X □

Theorem 2.0.10 *Given a subbasis for X, X is compact $\Leftrightarrow$ every open cover of X by sets from the subbasis has a finite subcover.*

$\Rightarrow$ Obvious

$\Leftarrow$ Consider the basis formed by taking finite intersections of sets in $\{U_\alpha\}_{\alpha \in I}$. By Proposition 2.0.9, it suffices to show that any open cover by sets in this basis has a finite subcover.

Let $\{V_\alpha\}$ be such an open cover. So WLOG each V_β is a finite intersection of sets from $\{U_\alpha\}$.

Suppose $\{V_\beta\}_{\beta \in J}$ has no finite subcover.

Well-order I and J.

Define $f : J \to I$ as follows so that for each β, $V_\beta \subset U_{f(\beta)}$ and $\{U_{f(\gamma)}\}_{\gamma \leq \beta} \cup \{V_\gamma\}_{\gamma > \beta}$ has no finite subcover.

Step 1: Define $f(j_0)$:

Write $V_{j_0} = U_{\sigma_1} \cap U_{\sigma_2} \cap \cdots \cap U_{\sigma_n}$.

Claim 1: For some $i = 1, \ldots, n$, $\{U_{\sigma_i}\} \cup \{V_\gamma\}_{\gamma > j_0}$ has no finite subcover.
Proof: Suppose not. Then $\exists$ a finite collection of the V_γ s.t. $\forall i$ $X = U_{\sigma_i} \cup V_{\gamma_1} \cup \cdots \cup V_{\gamma_r}$.

So

$$\begin{aligned} X &= \cap_{i=1}^n U_{\sigma_i} \cup V_{\gamma_1} \cup \cdots \cup V_{\gamma_r} \\ &= (\cap_{i=1}^n U_{\sigma_i}) \cup V_{\gamma_1} \cup \cdots \cup V_{\gamma_r} \\ &= V_{j_0} \cup V_{\gamma_1} \cup \cdots \cup V_{\gamma_r}. \end{aligned}$$

This contradicts our earlier assertion that X does not have a finite subcover by a finite collection of the V_γ. □

Choose i such that $\{U_{\sigma_i}\} \cup \{V_\gamma\}_{\gamma > j_0}$ has no finite subcover, and define

$$f(j_0) = \sigma_i. \tag{2.1}$$

Suppose now that f has been defined for all $\gamma < \beta$.
Claim 2:

$$\{U_{f(\gamma)}\}_{\gamma < \beta} \cup \{V_\gamma\}_{\gamma \geq \beta}$$

has no finite subcover.
Proof: Such a subcover would contradict the definition of $f(\hat{\gamma})$ where $\hat{\gamma}$ is the largest index occurring in the sets $\{U_{f(\gamma)}\}$ used in the subcover.

In other words, if $U_{f(\beta_1)}, \ldots, U_{f(\beta_k)}, V_{\beta_1'}, \ldots, V_{\beta_r'}$ is a subcover then it is also a subcover of $\{U_{f(\gamma)}\}_{\gamma \leq \beta_k} \cup \{V_\gamma\}_{\gamma > \beta_k}\}$. This contradicts the definition of $f(\hat{\gamma})$ where $\hat{\gamma} = \beta_k$.

Write $V_\beta = U_{\sigma_1} \cap \cdots \cap U_{\sigma_n}$.
Claim 3. For some $i = 1, \ldots, n$ $\{U_{f(\gamma)}\}_{\gamma < \beta} \cup \{U_{\sigma_i}\} \cup \{V_\gamma\}_{\gamma > \beta}$ has no finite subcover.
Proof: If not, we get a contradiction to the previous claim as in the proof of the definition of $f(j_0)$.

So choose i as in the previous claim and set $f(\beta) = \sigma_i$.

Now that f has been defined,
Claim 4. $\{U_{f(\beta)}\}$ has no finite subcover.
Proof: If $U_{f(\beta_1)} \cup \cdots \cup U_{f(\beta_k)}$ is a subcover then it is also a subcover of $\{U_{f(\gamma)}\}_{\gamma \leq \beta_k} \cup \{V_\gamma\}_{\gamma > \beta_k}$, contradicting the definition of $f(\beta_k)$.

But Claim 4 contradicts the definition of $\{U_\alpha\}$.

So $\{V_\beta\}_{\beta \in J}$ has a finite subcover and thus X is compact.

□

Theorem 2.0.11 [Tychonoff] *If X_α is compact for all α then $\prod_{\alpha \in I} X_\alpha$ is compact.*

Proof: Sets of the form

$$V_\alpha = U_\alpha \times \prod_{\gamma \neq \alpha} X_\gamma$$

(with U_α open in X_α) form a subbasis for the topology of X.

Let $\{V_\beta\}_{\beta\in J}$ be an open cover of X by sets in this subbasis.
Suppose $\{V_\beta\}$ has no finite subcover.
Let $F_\beta = (V_\beta)^c$.
Then

$$\cap_\beta F_\beta = \emptyset \tag{2.2}$$

but

$$\cap\{\text{any finite subcollection} F_\beta\} \neq \emptyset \tag{2.3}$$

where $V_\beta = U_{\alpha_0} \times \prod_{\gamma\neq\alpha_0} X_\gamma$

Note that for any β, the image of each of the projections of F_β is closed. That is, if $V_\beta = U_{\alpha_0} \times \prod_{\gamma\neq\alpha_0} X_\gamma$ then $\pi_{\alpha_0} F_\beta = (\pi_{\alpha_0} V_\beta)^c$ which is closed and for all other α, $\pi_\alpha F_\beta = X_\alpha$ which is closed.

So for any α, if $\cap_\beta(\pi_\alpha F_\beta) = \emptyset$ then $\pi_\alpha F_{\beta_1} \cap \cdots \cap \pi_\alpha(F_{\beta_r}) = \emptyset$ for some $\beta_1, \ldots, \beta_r$, since X_α is compact. This implies $F_{\beta_1} \cap \cdots \cap F_{\beta_r} = \emptyset$. This is a contradiction to (2.3). So there exists an $x_\alpha \in \cap_\beta \Pi_\alpha F_\beta$.

This is true for all α. So let $x = (x_\alpha)$.

Then $x \in \cap_\beta F_\beta$. This contradicts (2.2).

So $\{V_\beta\}$ has a finite subcover. Hence X is compact.

□