9.9 Cohomology

Definition 9.9.1 *A* cochain complex (C, d) *of abelian groups consists of an abelian group* C^p *for each integer* p *together with a morphism* $d^p : C^p \to C^{p-1}$ *for each* p *such that* $d^{p+1} \circ d^p = 0$. *The maps* d^p *are called* coboundary operators *or* differentials.

Aside frm the fact that we have chosen to number the groups differently, the concept of cochain complex is identical to that of chain complex. (Given a cochain complex (C, d) we could make it into a chain complex by renumbering the groups, letting $C_p := C^{-p}$, and vice versa.) So we can make all the same homological definitions and get the same homological theorems. A summary follows:

$\ker d^{p+1} : C^p \to C^{p+1}$ is denoted $Z^p(C)$. Its elements are called *cocycles.*

$\operatorname{Im} d^p : C^{p-1} \to C^p$ is denoted $B^p(C)$. Its elements are called *coboundaries.*

$H^p(C) := Z^p(C)/B^p(C)$ called the *pth cohomology group* of C.

A *cochain map* $f : C \to D$ consists of a group homomorphism f^p for each p s.t.

$$\begin{array}{ccc} C^p & \xrightarrow{d^{p+1}} & C^{p+1} \\ \downarrow f^p & & \downarrow f^{p+1} \\ D^p & \xrightarrow{d^{p+1}} & D^{p+1} \end{array} \qquad \text{commutes.}$$

Proposition 9.9.2

A cochain map f *induces a homomorphism denoted* $f^* : H^*(C) \to H^*(D)$. □

Theorem 9.9.3 *Let* $0 \to P \to Q \to R \to)$ *be a short exact sequence of chain complexes. Then there is an induced natural (long) exact cohomology sequence*

$$\ldots \to H^n(P) \to H^n(Q) \to H^n(R) \xrightarrow{\delta} H^{n+1}(P) \to H^{n+1}(Q) \to \ldots$$

Let (C, ∂) be a chain complex. Form a cochain complex (Q, δ) as follows.

$Q^p := \operatorname{Hom}(C_p, \mathbb{Z})$.

Notation: for $c \in C_p$, $f \in Q^p = \operatorname{Hom}(C, \mathbb{Z})$ write $\langle f, c\rangle$ for $f(c)$.

Define $\delta : Q^p \to Q^{p+1}$ bu $\langle \delta f, c\rangle := (-1)^{p+1}\langle f, \partial c\rangle$ where $c \in C_{p+1}$.

$\partial^2 = 0$ implies $\delta^2 = 0$.

Remark 9.9.4 *Changing one or more boundary maps by minus signs has no affect on kernels or images so it does not affect homology. The sign convention* $(-1)^{p+1}$ *chosen above makes* *the signs come out better in some of the later formulas. This is the convention used in Dold, Milner, Mac Lane, and Selick. An explanation of the intuition behind it can be found in Dold (page 173) or Selick (page 30). Notice Dold's convention on page 167 chosen so that when $n = 0$, $\partial f = 0$ implies f is a chain map. There are also other sign conventions ($(-1)^p$ or no sign at all) in the literature (e.g. Greenberg-Harper, Eilenberg-Steenrod, Munkres, Spanier, Whitehead) but they lead to less aesthetic formulas in several places and/or diagrams which only commute up to sign.*

Let $[c]$ and $[f]$ be homology and cohomology classes in C_*, Q_* respectively. Then $\langle [f], [c]\rangle$ has a well-defined meaning since if c' is another representative for c then for some d, $\langle f, c' - c\rangle = \langle f, \partial d\rangle = \pm\langle \delta f, d\rangle \pm \langle 0, d\rangle = 0$ and similarly if $f - f' = \delta g$ for some g then $\langle f - f', c\rangle = \langle \delta g, c\rangle = \pm\langle g, \partial c\rangle = 0$

$\langle\ ,\ \rangle$ is often called the *Kronecker product* or *Kronecker pairing.*

Any chain map $\phi : C \to D$ induces, by duality, a cochain map $\phi^* : \mathrm{Hom}(D, \mathbb{Z}) \to \mathrm{Hom}(C, \mathbb{Z})$. $\langle \phi^p(g), c\rangle := \langle g, \phi_p c\rangle$.

If C is a free chain complex (i.e. C_p is a free abelian group $\forall p$) then there is a formula, called the "Universal Coefficient Theorem" giving $H^*\big(\mathrm{Hom}(C, \mathbb{Z})\big)$ in terms of $H_*C()$. An immediate corollary of the Universal Coefficient Theorem is that if C, D are free chain complexes and $\phi : C \to D$ s.t. $\phi_* : H_p(C) \to H_p(D)$ is an isomorphism $\forall p$, then $\phi^* : H^p\big(\mathrm{Hom}(D, \mathbb{Z})\big) \to H^p\big(\mathrm{Hom}(C, \mathbb{Z})\big)$ is an isomorphism $\forall p$. We will not get to the Universal Coefficient Theorem in this course but we will give a direct proof of this corollary now.

From algebra recall:

Theorem 9.9.5 *If R is a PID and M is a free R-module than any R-submodule of M is a free R-module. In particular: letting $R = \mathbb{Z}$: A subgroup of a free abelian group is a free abelian group.* □

Proposition 9.9.6 *Let C be a free chain complex s.t. $H_q(C) = 0\ \forall\ q$. Then $H^q\big(\mathrm{Hom}(C, \mathbb{Z})\big) = 0\ \forall\ q$.*

Proof: $C_p / \ker \partial_p \cong \mathrm{Im}\, \partial_p = B_{p-1}$.

Since $H_*(C) = 0$, $\ker \partial_p = \mathrm{Im}\, \partial_{p+1} = B_p$. That is, $0 \to B_p \to C_p \xrightarrow{\partial_p} B_{p-1} \to 0$ is a short exact sequence. Since $B_{p-1} \subset C_{p-1}$ is a free abelian group, the sequence splits: $0 \to B_p \to C_p \underset{s}{\overset{\partial_p}{\rightleftarrows}} B_{p-1} \to 0$. i.e. $\exists$ a subgroup $U_p := \mathrm{Im}\, s$ of C_p s.t. $\partial U_p \cong B_{p-1}$ and $C_p \cong B_p \oplus U_p$ with $\partial(b, u) = (\partial u, 0)$.

$$C \qquad \xrightarrow{\ \partial\ } (B_{p+1} \oplus U_{p+1}) \longrightarrow (B_p \oplus U_p) \longrightarrow (B_{p-1} \oplus U_{p-1}) \longrightarrow$$

so dualizing gives a similar picture in $\text{Hom}(C, \mathbb{Z})$. That is, letting $U^p := \text{Hom}(U^p, \mathbb{Z})$ and $V^p := \text{Hom}(B^p, \mathbb{Z})$:

$\text{Hom}(C, \mathbb{Z})$

$$\to (U^{p-1} \oplus V_{p-1}) \longrightarrow (U^p \oplus V^p) \longrightarrow (U^{p+1} \oplus V_{p+1}) \longrightarrow$$

So $H^*\big(\text{Hom}(C, \mathbb{Z}\big) = 0$. $\square$

Corollary 9.9.7

Let $0 \to C \xrightarrow{\phi} D \xrightarrow{\alpha} E \to 0$ be a short exact sequence of chain complexes. Suppose that E is a free chain complex. If $\phi_* : H_q(C) \to H_q(D)$ is an isomorphism $\forall q$ then so is $\phi^* : H^*\big(\text{Hom}(D, \mathbb{Z}\big) \to H^*\big(\text{Hom}(C, \mathbb{Z}\big) = 0.$.

Proof: Since E_p is free $\forall p$, $D_p \cong C_p \oplus E_p$ and thus

$\text{Hom}(D_p, \mathbb{Z}) \cong \text{Hom}(C_p, \mathbb{Z}) \oplus \text{Hom}(E_p, \mathbb{Z})$. Thus in particular,

$0 \to \text{Hom}(E, \mathbb{Z}) \xrightarrow{\alpha^*} \text{Hom}(D, \mathbb{Z}) \xrightarrow{\phi^*} \text{Hom}(C, \mathbb{Z}) \to 0$ is again exact (a short exact sequence of cochain complexes). To show that ϕ^* is an isomorphism on cohomology, by the long exact sequence it suffices to show that $\text{Hom}(E, \mathbb{Z}) \cong 0$ $\forall q$. But $H_q(E) = 0$ $\forall q$ by the long exact homology sequence of $0 \to C \xrightarrow{\phi} D \xrightarrow{\alpha} E \to 0$ so the corollary follows from the previous proposition. $\square$

Note: The hypothesis that E be free is really needed. $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/(2\mathbb{Z}) \to 0$ is short exact but

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}(\mathbb{Z}/(2\mathbb{Z}) & \longrightarrow & \text{Hom}(\mathbb{Z}, \mathbb{Z}) & \longrightarrow & \text{Hom}(\mathbb{Z}, \mathbb{Z}) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & 0 & & \mathbb{Z} & & \mathbb{Z} & & \end{array}$$

is not.

Theorem 9.9.8 *(Algebraic Mapping Cylinder) Let C, D be free chain complexes and let $\phi : C \to D$. Then $\exists$ an injective chain homotopy equivalence $j : D \underset{k}{\overset{\simeq}{\rightleftarrows}} \tilde{D}$ (with chain homotopy inverse k) and an injection $i : C \to \tilde{D}$ s.t. $\phi = k \circ i$, $i \simeq j \circ \phi$, and $\tilde{D}/\operatorname{Im} j$ is free, and $\tilde{D}/\operatorname{Im} i$ is free.*

Corollary 9.9.9 *Let C, D be free chain complexes. Suppose $\phi^* C \to D$ such that $\phi_q : H_q(C) \to H_q(D)$ is an isomorphism $\forall q$. Then $\phi^* : H^q\big(\text{Hom}(D, \mathbb{Z})\big) \to H^q\big(\text{Hom}(C, \mathbb{Z})\big)$ is an isomorphism $\forall q$.*

Warning: To use this theorem to conclude that ϕ^p is an isomorphism for some particular p, we must know that ϕ_q is an isomorphism $\forall q$, not just for $q = p$. However it will follow from the Universal Coefficient Theorem that it is sufficient to know that ϕ_p and ϕ_{p-1} are isomorphisms to conclude that ϕ^p is an isomorphism.

Proof of Corollary (given Theorem.):

Previous lemma applied to $0 \to D \xrightarrow{j} \tilde{D} \to (\tilde{D}/\operatorname{Im} j) \to 0$ shows j^q is an isomorphism $\forall q$, which implies that $(\phi \circ j)_*$ is an isomorphism, which implies that i_* is an isomorphism. (Exercise: $f \simeq g \Rightarrow f^* \simeq g^*$.) Applying the lemma to $0 \to C \xrightarrow{i} \tilde{D} \to (\tilde{D}/\operatorname{Im} i) \to 0$ shows that i^q is an isomorphism $\forall q$. Therefore ϕ^q is an isomorphism $\forall q$. $\square$

9.9.1 Digression: Mapping Cylinders

Let $f : X \to Y$. If f is an injection then $\exists$ relative homology groups $H_*(Y, X)$ which "measure the difference" between $H_*(X)$ and $H_*(Y)$ and this is often convenient. What if f is not an injection? Then we can replace Y by a homotopy equivalent but "larger" space $\tilde{Y}$, called the mapping cylinder of f, such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"] & Y \arrow[d, "j"', "\simeq"] \\ & Y \end{tikzcd}$$

homotopy commutes ($j \circ f \simeq i$) with i an injection. The construction is as follows: $\tilde{Y} := (X \times I) \cup_{f'} Y$ where $f' : X \times \{0\} \to Y$ by $(a, 0) \mapsto f(x)$.

$X \hookrightarrow \tilde{Y}$ by $x \mapsto (x, 1)$. $\tilde{Y}$ can be "homotoped" to Y by squashing the cylinder.

Proof of Theorem 9.9.8:

9.9.2 Simplicial, Singular, and Cellular Cohomology

For a simplicial complex K we define the simplicial cochain complex of K by $C^*(K) := \operatorname{Hom}\big(C_*(K), \mathbb{Z}\big)$. Its cohomology is written $H^*(K)$ and called the simplicial cohomology of K.

For a topological space X we define its singular cohomology by $H^*(X) := H^*\big(S^*(X)\big)$ where $S^*X := \operatorname{Hom}\big(S_*(X), \mathbb{Z}\big)$.

And for a CW-comples, its cellular cohomology is defined as $H^*\big(D^*(X)\big)$ where $D^*X := \operatorname{Hom}\big(D_*(X), \mathbb{Z}\big)$.

From the isomorphisms on homology we get immediately $H^*(X) = H^*(|K|)$ and $H^*\big(D^*(X)\big) \cong H^*(X)$.

Can similarly define relative and reduced cohomology groups. e.g.

$H^*(X,A) := H^*\big(S^*(X,A)\big)$ where $S^*(X,A) := \mathrm{Hom}\big(S_*(X,A), \mathbb{Z}\big)$

Definition 9.9.10 (Eilenberg-Steenrod) *Let $\mathcal{A}$ be a class of topological pairs such that:*

1) (X,A) *in* $\mathcal{A} \Rightarrow (X,X)$, $(X,\emptyset)$, (A,A), $(A,\emptyset)$, *and* $(X \times I, A \times I)$ *are in* $\mathcal{A}$*;*

2) $(*,\emptyset)$ *is in* $\mathcal{A}$

A cohomology theory *on $\mathcal{A}$ consists of:*

E1) an abelian group $H^n(X,A)$ for each pair (X,A) in $\mathcal{A}$ and each integer n;

E2) a homomorphism $f^* : H^n(Y,B) \to H^n(X,A)$ for each map of pairs $f : (X,A) \to (Y,B)$;

E3) a homomorphism $\delta : H^n(X,A) \to H^{n+1}(A)$ for each integer n

such that:

A1) $1_* = 1$*;*

A2) $(gf)^* = f^* g^*$*;*

A3) δ is natural. That is, given $f : (X,A) \to (Y,B)$, the diagram

$$\begin{tikzcd}
H^n(B) \arrow[r, "(f \mid_A)^*"] \arrow[d, "\delta"] & H^n(A) \arrow[d, "\delta"] \\
H_{n+1}(Y,B) \arrow[r, "f^*"] & H_{n+1}(X,A)
\end{tikzcd}$$

commutes;

A4) Exactness:

$$\longrightarrow H^{n-1}(A) \longrightarrow H^n(X,A) \longrightarrow H^n(X) \longrightarrow H^n(A) \longrightarrow H^{n+1}(X,A) \longrightarrow$$

is exact for every pair (X,A) in $\mathcal{A}$

A5) Homotopy: $f \simeq g \Rightarrow f^* = g^*$.

A6) Excision: If (X, A) *is in* $\mathcal{A}$ *and* U *is an open subset of* X *such that* $\overline{U} \subset \overset{\circ}{A}$ *and* $(X - U, A - U)$ *is in* $\mathcal{A}$ *then the inclusion map* $(X \setminus U, A \setminus U) \to (X, A)$ *induces an isomorphism* $H^n(X, A) \xrightarrow{\cong} H^n(X \setminus U, A \setminus U)$ *for all* n*;*

A7) Dimension: $H^n(*) = \begin{cases} \mathbb{Z} & \text{if } n = 0; \\ 0 & \text{if } n \neq 0. \end{cases}$

Theorem 9.9.11 *Singular cohomology is a cohomology theory.*

Proof: For exactness, observe that because all the complexes are free, the fact that $0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$ is exact (and thus $S_*(X) \cong S_*(A) \oplus S_*(X, A)$) implies that $0 \to S^*(X, A) \to S^*(X) \to S^*(A) \to 0$ is exact. Everything else is immediate from the previous theorem and the corresponding statment for homology (and, of course, we get the slightly stronger version of excision, not requiring that U be open, since singular homology satisfies that).

The following theorems also follow easily from the homological counterparts:

Theorem 9.9.12 *(Mayer-Vietoris): Suppose that* $(X_1, A) \xrightarrow{j} (X, X_2)$ *induces an isomorphism on cohomology. (e.g. if* X_1 *and* X_2 *are open. Then there is a long exact cohomology sequence*

$$\ldots \to H_{n-1}(A) \xrightarrow{\Delta} H^n(X) \to H^n(X_1) \oplus H^n(X_2) \to H^n(A) \xrightarrow{\Delta} H^{n+1}(X) \to \ldots$$ □

Theorem 9.9.13

$$H^n(X) \cong \begin{cases} \tilde{H}^n(X) & n > 0; \\ \tilde{H}^0(X) \oplus \mathbb{Z} & n = 0. \end{cases}$$

Also $\tilde{H}^q(X) \cong H^q(X, *)$ □

Theorem 9.9.14 $H^q(S^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & q \neq 0, n \end{cases}$

Proof: Use celluar cohomology. Write $S^{=}e^0 \cup e^n$.

$$D_*(S^n) \qquad \underset{}{0} \longrightarrow \underset{nth\text{ pos.}}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow \underset{0th\text{ pos.}}{\mathbb{Z}} \longrightarrow 0$$

$$D^*(S^n) \qquad \underset{0th\text{ pos.}}{0 \longrightarrow \mathbb{Z}} \longrightarrow 0 \longrightarrow \ldots \longrightarrow \underset{nth\text{ pos.}}{0 \longrightarrow \mathbb{Z}} \longrightarrow 0$$ □

Theorem 9.9.15

n even:

$$H^q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/(2\mathbb{Z}) & q \text{ even, } q < n \\ 0 & q \text{ odd or } q > n \end{cases}$$

n odd:

$$H^q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ \mathbb{Z}/(2\mathbb{Z}) & q \text{ even, } q < n \\ 0 & q \text{ odd or } q > n. \end{cases}$$

□

Theorem 9.9.16

$$H^q(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & q \text{ even, } q \leq 2n \\ 0 & q \text{ odd, } q > 2n. \end{cases}$$

$$H^q(\mathbb{H}P^n) = \begin{cases} \mathbb{Z} & q \equiv 0(4); \\ 0 & q \not\equiv 0(4). \end{cases}$$

□

Proof: Write $\mathbb{C}P^n = e^0 \cup e^2 \cup \ldots e^{2n}$. Write $\mathbb{H}P^n = e^0 \cup e^4 \cup \ldots e^{4n}$. □