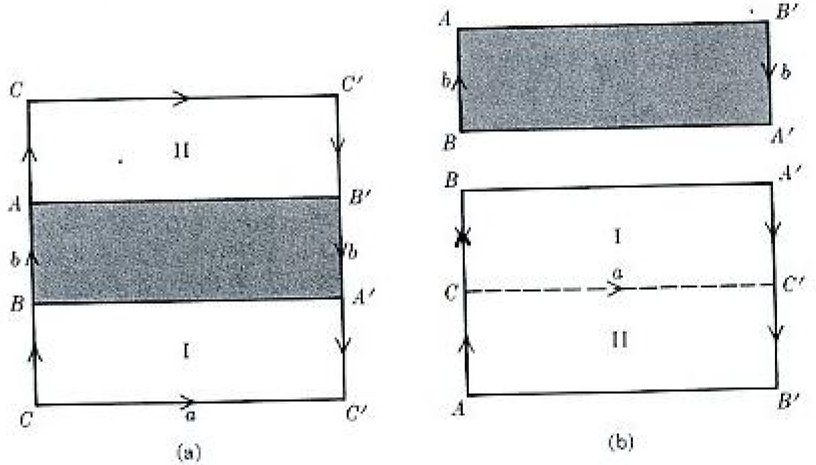

FIGURE 1.5 The Klein bottle is the union of two Möbius strips.

It is clear from our definitions that there is no distinction between $S_1 \# S_2$ and $S_2 \# S_1$; i.e., the operation is commutative. It is not difficult to see that the manifolds $(S_1 \# S_2) \# S_3$ and $S_1 \# (S_2 \# S_3)$ are homeomorphic. Thus, we see that the connected sum is a commutative, associative operation on the set of homeomorphism types of compact surfaces. Moreover, Example 4.1 shows the sphere is a unit or neutral element for this operation. We must not jump to the conclusion that the set of homeomorphism classes of compact surfaces forms a group under this operation: There are no inverses. It only forms what is called a semigroup.

The connected sum of two orientable manifolds is again orientable. On the other hand, if either S_1 or S_2 is nonorientable, then so is $S_1 \# S_2$.

5 Statement of the classification theorem for compact surfaces

In the preceding section we have seen how examples of compact surfaces can be constructed by forming connected sums of various numbers of tori and/or projective planes. Our main theorem asserts that these examples exhaust all the possibilities. In fact, it is even a slightly stronger statement, in that we do not need to consider surfaces that are connected sums of both tori and projective planes.

Theorem 5.1 *Any compact surface is either homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes.*

Second step. Elimination of adjacent edges of the first kind. We have now obtained a polygon D whose edges have to be identified in pairs to obtain the given surface S. These identifications may be indicated by the appropriate symbol; e.g., in Figure 1.16, the identifications are described by

$$aa^{-1}fbb^{-1}f^{-1}e^{-1}gcc^{-1}g^{-1}dd^{-1}e.$$

If the letter designating a certain pair of edges occurs with *both* exponents, $+1$ and -1, in the symbol, then we will call that pair of edges a pair of the *first kind;* otherwise, the pair is of the *second kind.* For example, in Figure 1.16, all seven pairs are of the first kind.

We wish to show that an adjacent pair of edges of the first kind can be eliminated, provided there are at least four edges in all. This is easily seen from the sequence of diagrams in Figure 1.17. We can continue this process until all such pairs are eliminated, or until we obtain a polygon with only two sides. In the latter case, this polygon, whose symbol will be aa or aa^{-1}, must be a projective plane or a sphere, and we have completed the proof. Otherwise, we proceed as follows.

Third step. Transformation to a polygon such that all vertices must be identified to a single vertex. Although the edges of our polygon must be

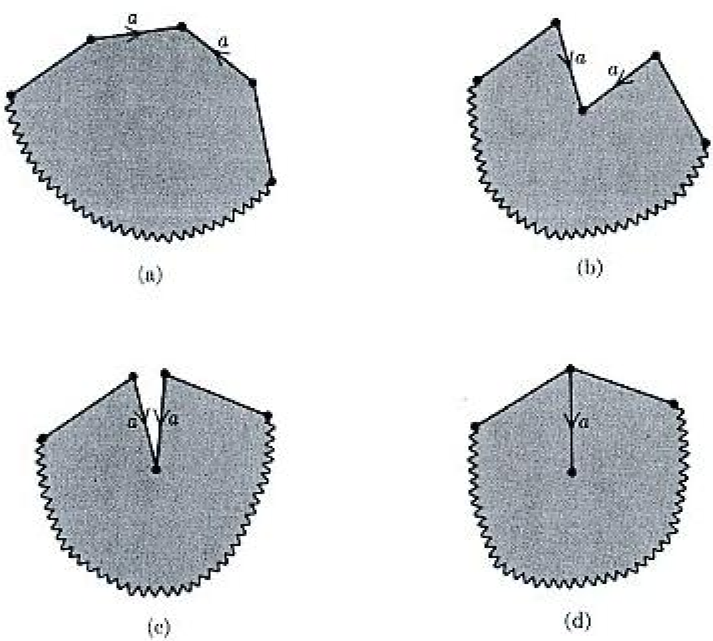

FIGURE 1.17 Elimination of an adjacent pair of edges of the first kind.

identified in pairs, the vertices may be identified in sets of one, two, three, four, ... Let us call two vertices of the polygon *equivalent* if and only if they are to be identified. For example, the reader can easily verify that in Figure 1.16 there are eight different equivalence classes of vertices. Some equivalence classes contain only one vertex, whereas other classes contain two or three vertices.

Assume we have carried out step two as far as possible. We wish to prove we can transform our polygon into another polygon with all its vertices belonging to one equivalence class.

Suppose there are at least two different equivalence classes of vertices. Then, the polygon must have an adjacent pair of vertices which are nonequivalent. Label these vertices P and Q. Figure 1.18 shows how to proceed. As P and Q are nonequivalent, and we have carried out step two, it follows that sides a and b are *not* to be identified. Make a cut along the line labeled c, from the vertex labeled Q to the other vertex of the edge a (i.e., to the vertex of edge a, which is distinct from P). Then, glue the two edges labeled a together. A new polygon with one less vertex in the equivalence class of P and one more vertex in the equivalence class of Q results. If possible, perform step two again. Then carry out step three to reduce the number of vertices in the equivalence class of P still further, then do step two again. Continue alternately doing step three and step two until the equivalence class of P is eliminated entirely. If more than one equivalence class of vertices remains, we can repeat this procedure to reduce the number by one. If we continue in this manner, we ultimately obtain a polygon such that all the vertices are to be identified to a single vertex.

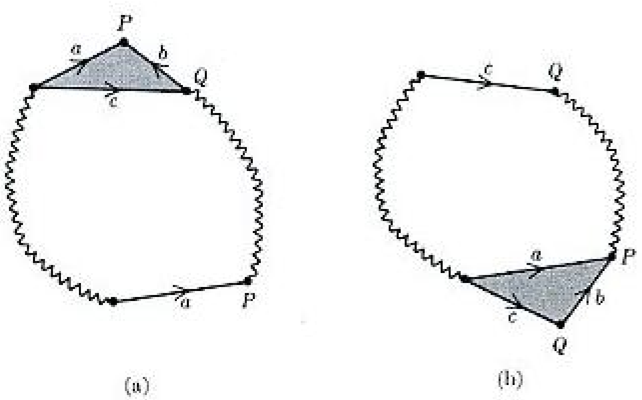

FIGURE 1.18 Third step in the proof of Theorem 5.1.

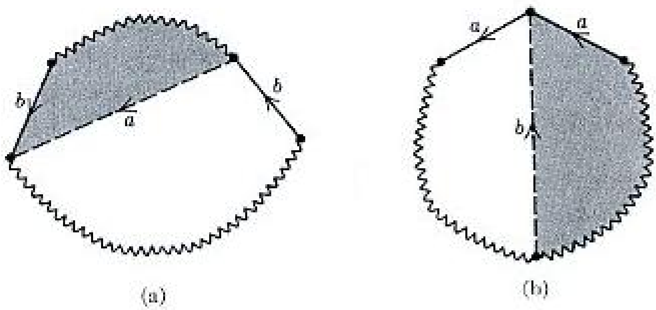

FIGURE 1.19 Fourth step in the proof of Theorem 5.1.

Fourth step. How to make any pair of edges of the second kind adjacent. We wish to show that our surface can be transformed so that any pair of edges of the second kind are adjacent to each other. Suppose we have a pair of edges of the second kind which are nonadjacent, as in Figure 1.19(a). Cut along the dotted line labeled a and paste together along b. As shown in Figure 1.19(b), the two edges are now adjacent.

Continue this process until all pairs of edges of the second kind are adjacent. If there are no pairs of the first kind, we are finished, because the symbol of the polygon must then be of the form $a_1a_1a_2a_2 \ldots a_na_n$, and hence S is the connected sum of n projective planes.

Assume to the contrary that at this stage there is at least one pair of edges of the first kind, each of which is labeled with the letter c. Then we assert that there is at least one other pair of edges of the first kind such that these two pairs separate one another; i.e., edges from the two pairs occur alternately as we proceed around the boundary of the polygon (hence, the symbol must be of the form $c \ldots d \ldots c^{-1} \ldots d^{-1} \ldots$, where the dots denote the possible occurrence of other letters).

To prove this assertion, assume that the edges labeled c are not separated by any other pair of the first kind. Then our polygon has the appearance indicated in Figure 1.20. Here A and B each designate a whole sequence of edges. The important point is that any edge in A must be identified with another edge in A, and similarly for B. No edge in A is to be identified with an edge in B. But this contradicts the fact that the initial and final vertices of either edge labeled "c" are to be identified, in view of step number three.

Fifth step. Pairs of the first kind. Suppose, then, that we have two pairs of the first kind which separate each other as described (see Figure 1.21). We shall show that we can transform the polygon so that the four sides in question are consecutive around the perimeter of the polygon.

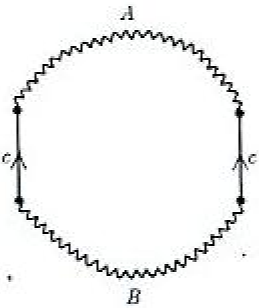

FIGURE 1.20 A pair of edges of the first kind.

First, cut along c and paste together along b to obtain Figure 1.21(b). Then, cut along d and paste together along a to obtain (c), as desired.

Continue this process until all pairs of the first kind are in adjacent groups of four, as $cdc^{-1}d^{-1}$ in Figure 1.21(c). If there are no pairs of the

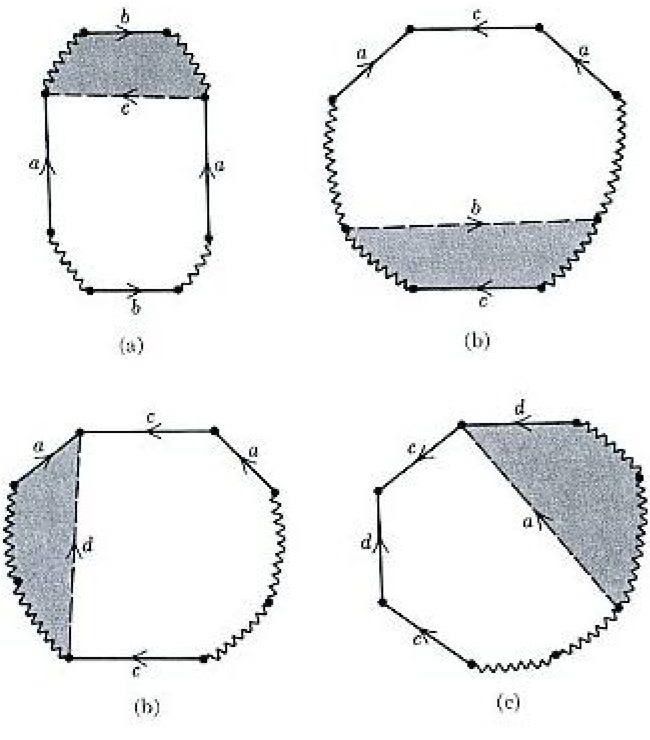

FIGURE 1.21 Fifth step in the proof of Theorem 5.1.

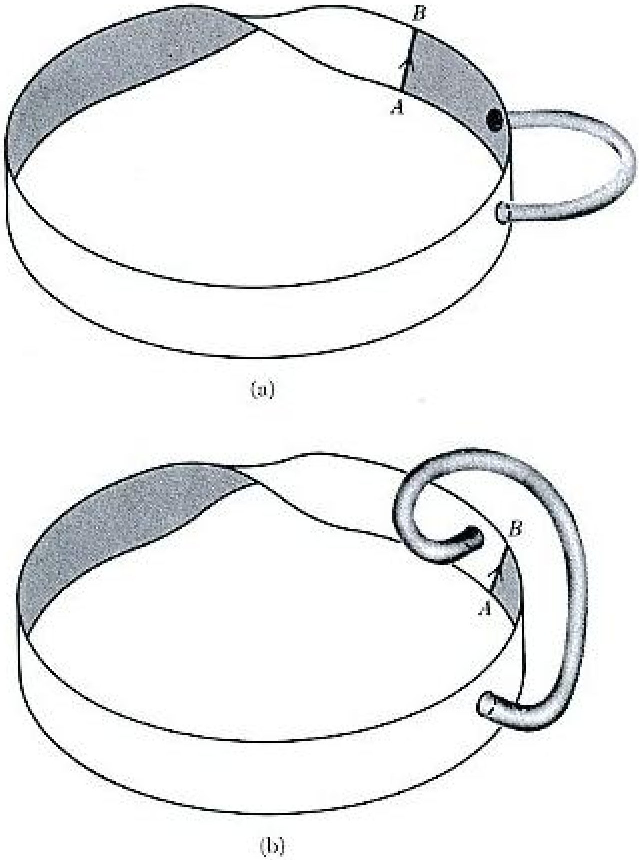

FIGURE 1.23 (a) Connected sum of a Möbius strip and a torus. (b) Connected sum of a Möbius strip and a Klein bottle.

cut in it. In the second stage we then connect the boundaries of these two holes with a tube that is the remainder of the torus or Klein Bottle. The difference between the two cases depends on whether we connect the boundaries so they will have the same or opposite orientations. This is illustrated in Figure 1.23, where S is a Möbius strip.

We now assert that the two spaces shown in Figure 1.23(a) and (b) (i.e., the connected sum of a Möbius strip with a torus and a Klein Bottle, respectively) are homeomorphic. To see this, imagine that we cut each of these topological spaces along the lines AB. In each case, the result is the connected sum of a rectangle and a torus, with the two ends of the