9.6 Homology of CW-complexes

Let X be a CW-complex.

If $T : \Delta_n \to X \in S_n(X)$ is a generator, then $\operatorname{Im} T$ is compact so $\operatorname{Im} T \subset X^{(p)}$ for some p. Therefore $S_*(X) = \cup_p S_*(X^{(p)})$.

How does this tell us $H_*(X)$ in terms of the $H_*(X^{(p)})$'s?

9.6.1 Direct Limits

Definition 9.6.1 *A partially ordered set J is called a* directed set *if $\forall i, j \in J$ $\exists k$ s.t. $i \leq k$ and $j \leq k$.*

Definition 9.6.2 *Given a directed set J, a* directed system *of abelian groups indexed by J consists of:*

1. *An abelian group G_j for each $j \in J$;*
2. *For each pair $i, j \in J$ a group homomorphism $\phi_{j,i} : G_i \to G_j$ s.t. $\phi_{j,j} = 1_{G_j}$ and $\phi_{k,j} \circ \phi_{j,i} = \phi_{k,i}$.*

Examples

1. $J = \mathbb{Z}^+$; $G_n = M_n(\mathbf{k})$ ($n \times n$ matrices over a field $\mathbf{k}$)

   $\phi_{ij} : M_i(\mathbf{k}) \to M_j(\mathbf{k})$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$.
2. $J = \{$finite subcomplexes of a CW complex X, ordered by inclusion$\}$

   $G_Y = H_p(Y)$ (where Y is a finite subcomplex of X)
3. X topological space; $J = \{$open subsets of X ordered by inclusion$\}$

   $G_U = H_p(U)$.
4. $J = \mathbb{Z}^+$; $G_n = \mathbb{Z}$; $\phi_{j,i} : \mathbb{Z} \to \mathbb{Z}$ by $1 \mapsto p^{j-i}$

Definition 9.6.3 *The* direct limit *of the direct system $\{G_j\}_{j \in J}$ consists of an abelian group G and homomorphisms $\phi_j : G_j \to G$ s.t.*

1.

$$\begin{tikzcd}
G_i \arrow[rr, "\phi_{j,i}"] \arrow[dr, "\phi_i"] & & G_j \arrow[dl, "\phi_j"] \\
& G &
\end{tikzcd}$$

commutes $\forall i, j$

2. G is univesal w.r.t. property (1). i.e., given H and homomorphisms $\psi_j : G_j \to H$ s.t. $\psi_i \circ \phi_{j,i} = \psi_j$, $\exists!\, \theta : G \to H$ s.t. $\forall i, j$

$$\begin{tikzcd}
G_i \arrow[rr, "\phi_{j,i}"] \arrow[dr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & G_j \arrow[dl, "\phi_j"'] \arrow[ddl, "\psi_j"] \\
& G \arrow[d, dashed, "\theta"] & \\
& H &
\end{tikzcd}$$

We write $G = \varinjlim_J \{G_j\}$.
Note: By the usual categorical argument, a direct system has at most one direct limit up to isomorphism. As we shall see, every direct system of abelian groups has a direct limit.

Observe that if $\phi_{j,i}$ is an inclusion map $\forall i, j$ then $G = \cup_{j \in J} G_j$ is the direct limit of the system.

Theorem 9.6.4 *Every direct direct system of abelian groups has a direct limit.*

Proof: Let $H = \oplus_{j \in J} G_j$ with $\alpha_j : G_j \to H$ the canonical inclusion.

Let $G = H/\sim$ where $\alpha_i(g) \sim \alpha_j(g)$ $\forall i, j$ and $\forall g \in G_i$. More precisely, $G = H/H'$ where H' is the subgroup of H generated by $\{\alpha_i(g) - \alpha_j \phi_{j,i}(g)\}$.

Let $\pi : H \to G$ be the quotient map.

Define ϕ_j to be the composite $G_j \xrightarrow{\alpha_j} H \xrightarrow{\phi} G$.

Then $\forall i, j$ and $\forall g \in G$, $\phi_j \phi_{j,i}(g) = \pi \alpha_j \phi_{j,i}(g) = \pi \alpha_i(g) = \phi_i(g)$.

Also, given k and maps $\psi_j : G_j \to K$ s.t. $\psi_i \circ \phi_{j,i} = \psi_j$: The maps ψ_j induce a unique map $\theta : H \to K$ (by the universal property of direct sum). Furthermore, since $\psi_i \circ \phi_{j,i} = \psi_j$, $\theta\big|_{H'}$ is the trivial map so by the universal property of quotient

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "\pi"] & K \\
G \arrow[ur, dotted, "\theta"'] &
\end{tikzcd}$$

□

Remark 9.6.5 *The definitions make sense and this proof still works even if the poset J is not a direct system. There is a more general notion called* colimit *when the poset J is not directed.*

From now on we will omit the inclusion maps α_j.

Notice: Any element of G has a representative of the form $\phi_k(g)$ for some $g \in G_k$.
Proof: Let $X = (g_j)_{j \in J}$ represent an element of G. Since x has only finitely many nonzero components, the definition of direct system implies that $\exists k \in J$ s.t. $j \leq k \; \forall j$ s.t. $g_j \neq 0$. Then adding $\phi_{k,j}(g_j) - g_j$ to x for all j s.t. $g_j \neq 0$ gives a new representative for x with only one nonzero component. (i.e. for some k, $x = \phi_k(g)$ with $g \in G_k$.)

Lemma 9.6.6 *If $g \in G_k$ s.t. $\phi)k(g) = 0$ then $\phi_{m,k}(g) = 0$ for some m.*

Proof:
Notation: For "homogeneous" elements of $\oplus_{\alpha \in J} G_\alpha$ (i.e. elements with just 1 nonzero component) write $|h| = \alpha$ to mean that $h \in G_\alpha$, or more precisely that the only nonzero component of h lies in G_α.

$\phi_k(g) = 0 \Rightarrow g \in H' \Rightarrow$

$$g = \sum_{t=1}^{n} \phi_{j_t, i_t} g_t - g_t \qquad \text{where } g_t \in G_{i_t} \tag{9.1}$$

Find m s.t. $k \leq m$ and $i_r \leq m$ and $j_r \leq m \; \forall r$. Set $g' = \phi_{m,k} g$.

Adding $g' - g = \phi_{m.k} g - g$ to equation 9.1 gives

$$g' = \sum_{t+0}^{n} \phi_{j_t, i_t} g_t - g_t \qquad \text{where } g_0 = g \tag{9.2}$$

Note that for any $\alpha < m$, collecting terms on RHS in G_α gives 0, since LHS is 0 in degree α.

Among $S := \{i_0, \dots, i_n, j_0, \dots, j_n, m\}$ find α which is minimal. (i.e. each other index occuring is either greater or not comparable) Since $j_t\ i_t$, α is one of the i's so this means $J - t \neq \alpha$ for any t.

For each t with $|g_t| = \alpha$, add $g_t - \phi_{m,|g_t|} g_t$ to both sides of equation 9.2.

As noted above, $\sum_{\{t||g_t|=\alpha\}} g_t = 0$ so $\sum_{\{t||g_t|=\alpha\}} \phi_{m,|g_t|} g_t$ is also 0 and so we are actually adding 0 to the equation. However we can rewrite it using:

$\phi_{j_t,i_t} g_t - g_t + g_t - \phi_{m,|g_t|} g_t = \phi_{j_t,i_t} g_t - \phi_{m,|g_t|} g_t \overset{(|g_t|=i_t)}{=\!=\!=} \phi_{j_t,i_t} g_t - \phi_{m,j_t} \phi_{j_t,i_t} g_t = \phi_{m,j_t} \tilde{g}_t$ where $\tilde{g}_t = -\phi_{j_t,i_t} g_t$. Therefore we now have a new expression of the form $g' = \sum \phi_{\mathrm{J}_t,i_t} g)t - g_t$; however the new2 set S is smaller than before since it no longer contains α (and no new index was added).

Repeat this process until the set S consists of just $\{m\}$. Then no i's are left in S (since $i_t < m\ \forall t$) which means that there are no terms left in the sum. That is, Equation 9.1 reads $g' = 0$, as required. ☐

Notice that from the construction: If J is totally ordered and $\exists N$ s.t. $\phi_{n,k}$ is an isomorphism $\forall k, n \geq N$ (in which case we say the system *stabilizes*) then the direct limit is isomorphism to the "stable" group G_N.

Remark 9.6.7 *Above can be dualized by turning the arrows around: That is, define*

Inversely directed system = poset J s.t. $\forall k, n \in J\ \exists j \in J$ s.t. $j \leq k$, $j \leq n$.

Define an inverse system of abelian groups to be a collection of abelian groups G_j and "compatible" group homomorphisms $\phi_{k,j}$ indexed by the inverse system. The inverse limit, $\varprojlim_J G_j$, of the inverse system is defined as an abelian group which has the property that there exists a "compatible" collection of homomorphisms $\phi_k : \varprojlim_J G_j \to G_k$ and such that given any group H with the same properties $\exists!\ \theta : H \varprojlim_J G_j$ making the diagrams commute. The construction of a group satisfying this definition is given by $\varprojlim_J G_j = \{(x_j) \in \prod_{j \in J} G_j \mid \phi_{k,j} x_j = x_k\}$.

Theorem 9.6.8 *("Homology commutes with direct limits")*

Let $C = \varinjlim (C_j)_*$. Then $H(C) = \varinjlim_J H_*(C_j)$.

Remark 9.6.9 *Even if $\varinjlim_J C_j$ is just a union, $\{H_*(C_j)\}$ may be a non-trivial direct system. (Homology need not preserve monomorphisms.)*

Proof: Let $\psi_{j,i} : (C_i)_* \to (C_j)_*$ be the maps in the direct system $\varinjlim_J C_j$. Definition of maps

$\phi_{j,i} : H(C_{i*}) \to H(C_{j*})$ is $\phi_{j,i} = (\psi_{j,i})_*$.

$$\begin{tikzcd}
H(C_{i*}) \arrow[rr, "\phi_{j,i} = (\psi_{j,i})_*"] \arrow[dr, "\phi_i"] \arrow[ddr, "(\psi_i)_*"'] & & H(C_{j*}) \arrow[dl, "\phi_j"'] \arrow[ddl, "(\psi_j)_*"] \\
& \varinjlim_J H(C_j) \arrow[d, dashed, "\theta"] & \\
& H(C) &
\end{tikzcd}$$

Claim θ is onto:

Given $[x] \in H(C)$, where $x \in C$, find a representative $x_k \in C_{k*}$ for x. (That is, $x = \psi_k x_k$).

Since x represents a homology class, $\partial x = 0$. Hence $\psi_k \partial x_k = \partial \psi_k x_k = \partial x = 0$. Replacing x_k by $x_m = \phi_{m,k} x_k$ for some m, get a new representative for x s.t. $\partial x_m = 0$. Therefore x_m represents a homology class $[x_m] \in H(C_{m*})$ and

$$\begin{tikzcd}
{[x_m]} \arrow[dr] & H(C_{m*}) \arrow[rr] \arrow[dr] & & \varinjlim_J H(C_j) \arrow[dl, "\theta"] \\
& {[x]} & H(C) &
\end{tikzcd}$$

shows $\in \operatorname{Im} \theta$.

Claim θ is $1-1$:

Let $y \in \lim_J H(C_j)$ s.t. $\theta(y) = 0$.

Find a representative $[x_k] \in H(C_{k*})$ for y, where $x_k \in X_{k*}$. (That is, $y = \phi_k(x_k)$.)

$$\begin{tikzcd}
{[x_k]} \arrow[rr] & & {[y]} \\
H(C_{k*}) \arrow[rr] \arrow[dr, "\psi_{k*}"'] & & \varinjlim_J H(C_j) \arrow[dl, "\theta"] \\
& H(C) &
\end{tikzcd}$$

Since $\theta y = 0$, $[\psi_k x_k] = 0$ in $H(C)$. That is, $\exists v \in C$ s.t. $\partial v = \psi_k x_k$.

May choose l s.t. $v = \psi_{l*}(w_l)$.

Find m s.t. $k, l \leq m$. Then replacing x_k, w_l by their images in $(C_m)_*$ we get that $x - \partial w_m$ stabilizes to 0 so that $\exists m' \geq m$ s.t. $[x_{m'}] = [\partial w_{m'}] = 0$. Hence $y = 0$. $\square$

Theorem 9.6.10 $H_*(X) = \varinjlim_p H_*(X^{(p)})$

Proof: Every compact subset of X is contained in $X^{(N)}$ for some N, so by $A8$, $S_*(X) = \cup_p S(X^{(p)}) = \varinjlim_p S_*(X^{(p)})$. Therefore $H_*(X) = \varinjlim_p H_*(X^{(p)})$. $\square$

Theorem 9.6.11 *If $X = \cup_{n=1}^{\infty} V_n$ where V_n open in X and $V_n \subset V_{n+1}$ then $H_*(X) = \varinjlim_n H_*(V_n)$.*

Proof: Sufficient to show that $S_*(X) \cup_{n=1}^{\infty} S_*(V_n)$.

If $T \in S_*(X)$ is a generator then $\operatorname{Im} T$ is compact.

$\{V_n\}$ covers X so $\operatorname{Im} T \subset V_n$ for some n (since V_n's nested).

Hence $T \in S_*(V_n)$ for that n. $\square$

9.7 Cellular Homology

Let X be a CW-complex.

By convention $X^{(p)} = \emptyset$ if $p < 0$.

Let $D_p(X) = H_p(X^{(p)}, X^{(p-1)})$.

Define $\partial_D : D_p(X) \to D_{p-1}(X)$ to be the connecting homorphism from the exact sequence of the triple $(X^{(p)}, X^{(p-1)}, X^{(p-2)})$. Therefore ∂_D factors as

$H_p(X^{(p)}, X^{(p-1)}) \xrightarrow{\partial} H_p(X^{(p)}) \xrightarrow{j_*} H_p(X^{(p-1)}, X^{(p-2)})$.

Hence $\partial_D^2 = 0$ since

$$H_p(X^{(p)}, X^{(p-1)}) \xrightarrow{\partial} H_p(X^{(p)}) \xrightarrow{j_*} H_p(X^{(p-1)}, X^{(p-2)}) \xrightarrow{\partial} H_p(X^{(p-1)}) \xrightarrow{j_*} H_p(X^{(p-2)}, X^{(p-3)})$$

contains the consecutive maps $H_p(X^{(p)}) \xrightarrow{j_*} H_p(X^{(p-1)}, X^{(p-2)}) \xrightarrow{\partial} H_p(X^{(p-1)})$ which is 0 from the exact sequence of the pair $(X^{(p-1)}, X^{(p-2)})$.

Therefore $\big(D_*(X), \partial_D\big)$ forms a chain complex called the *cellular chain complex* of X. Its homology is called the *cellular homology* of X, written $H_*^{\text{cell}}(X)$.

Lemma 9.7.1 $H_q(X^{(p)}, X^{(p-1)}) \cong \begin{cases} \mathrm{F}_{\mathrm{ab}}\{p-\text{ cells of } X\} & q = p \\ 0 & \text{otherwise} \end{cases}$

Proof: In each p-cell of X, select a point x_j.

Notice that $X^{(p-1)} \cup (e_j^p - x_j) \simeq X^{(p-1)}$. That is, $X^{(p-1)} \cup (e_j^p - x_j)$ is the subspace of $X^{(p)}$ formed by attaching D^m to $X^{(p-1)}$ along ∂D^m. $X^{(p-1)} \cup (e_j^p - x_j)$ is formed by attaching $D^m - \{*\}$ to $X^{(p-1)}$ along ∂D^m. But using the homotopy equivalence $D^m - \{*\} \simeq \partial D^m$ can construct a continuous deformation of $X^{(p-1)} \cup (e_j^p - x_j)$ back to $X^{(p-1)}$. (i.e. gradually enlarge the hole.)

$$X^{(p-1)} \simeq X^{(p-1)} \cup \Bigg(\bigcup_{p-\text{cells of } X} (e_j^p \smallsetminus \{x_j\}) \Bigg)$$

Note: If $A \subset B \subset X$ where j is a homotopy equivalence than $H_*(X, A) \xrightarrow{\cong} H_*(B)$ using

$$\begin{tikzcd}
{} \arrow[r] & H_q(A) \arrow[r] \arrow[d, "\cong"] & H_q(B) \arrow[r] \arrow[d, equal] & H_q(X,A) \arrow[r] \arrow[d] & H_{q-1}(A) \arrow[r] \arrow[d, "\cong"] & H_{q-1}(X) \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & H_q(A) \arrow[r] & H_q(B) \arrow[r] & H_q(X,A) \arrow[r] & H_{q-1}(A) \arrow[r] & H_{q-1}(X) \arrow[r] & {}
\end{tikzcd}$$

and the 5-lemma. (This avoids using the homotopy axiom directly, which would require a homotopy equivalence of pairs.)

Therefore

$$H_*(X^{(p)}, X^{(p-1)}) \cong H_*\Bigg(X^{(p)}, X^{(p-1)} \cup \Big(\bigcup_{p-\text{cells of } X} (e_j^p \smallsetminus \{x_j\})\Big)\Bigg)$$

Notice that $X^{(p-1)} \cup \Big(\bigcup_{p-\text{cells of } X} (e_j^p - x_j)\Big) = X^{(p)} \smallsetminus (\cup\{x_j\})$ which is open.

By excision

$$\begin{aligned} H_*\Bigg(X^{(p)}, X^{(p-1)} \cup \Big(\bigcup_{p-\text{cells of } X} (e_j^p \smallsetminus \{x_j\})\Big)\Bigg) &\cong H_*\Bigg(\Big(\bigcup_{p-\text{cells of } X} (e_j^p)\Big), \Big(\bigcup_{p-\text{cells of } X} (e_j^p \smallsetminus \{x_j\})\Big)\Bigg) \\ &\cong \bigoplus_{p-\text{cells of } X} H_*\big(e_j^p, e_j^p \smallsetminus \{x_j\}\big) \end{aligned}$$

where we have excised the closed set $X^{(p-1)}$ from the open set $X^{(p-1)} \smallsetminus \big(\cup\{x_j\}\big)$.

Up to homeomorphism, $e_j^p = \mathring{D}^p$ and $H_q\left(\mathring{D}^p, \mathring{D}^p \smallsetminus\{*\}\right) = \begin{cases} \mathbb{Z} & q = p \\ 0 & \text{otherwise} \end{cases}$ since

$$\begin{tikzcd}
H_q(\mathring{D}^p \smallsetminus\{*\}) \arrow[r] & H_q(\mathring{D}^p) \arrow[r] \arrow[d, equal] & H_q(\mathring{D}^p, \mathring{D}^p \smallsetminus\{*\}) \arrow[r, "\cong"] & H_{q-1}(\mathring{D}^p - \{*\}) \arrow[r] \arrow[d, equal] & H_{q-1}(\mathring{D}^p) \arrow[r] & {} \\
 & 0 & & H_{q-1}(S^{p-1}) & &
\end{tikzcd}$$

Hence

$$H_q(X^{(p)}, X^{(p-1)}) = \begin{cases} \bigoplus_{p-\text{cells of } X} \mathbb{Z} & \text{if } q = p\,; \\ 0 & \text{otherwise} \end{cases} \cong \begin{cases} F_{\text{ab}}\{p\text{−cells of } X\} & \text{if } q = p\,; \\ 0 & \text{otherwise.} \end{cases}$$

□

Lemma 9.7.2 $H_q(X^{(n)} = \begin{cases} H_q(X) & q < n\,; \\ 0 & q > n. \end{cases}$

Proof:

If $q > n$:

$$\begin{tikzcd}
H_{q+1}(X^{(n)}) \arrow[r] & H_{q+1}(X^{(n)}, X^{(n-1)}) \arrow[r] \arrow[d, equal] & H_q(X^{(n-1)}) \arrow[r, "\cong"] & H_q(X^{(n)}) \arrow[r] & H_q(X^{(n)}, X^{(n-1)}) \arrow[d, equal] \\
 & 0 & & & 0
\end{tikzcd}$$

So $H_q(X^{(n)} \cong H_q(X^{(n-1)} \cong \ldots \cong H_q(X^{(-1)} \cong H_q(\emptyset) \cong 0$.

Similarly if $q < n$:

$$\begin{tikzcd}
H_{q+1}(X^{(n+1)}, X^{(n)}) \arrow[r] \arrow[d, equal] & H_{q+1}(X^{(n)}) \arrow[r, "\cong"] & H_q(X^{(n+1)}) \arrow[r] & H_q(X^{(n+1)}, X^{(n)}) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

So $H_q(X^{(n)} \cong H_q(X^{(n+1)} \cong \ldots \cong H_q(X^{(p)}\ \forall p$.

Therefore $H_q(X) \cong \varinjlim H_q(X^{(p)} = H_q(X^{(n)})$. ☐

Theorem 9.7.3 *$H\big(D_*(X)\big) = H_*(X)$.*

Proof: From the triples $(X^{(n+1)}, X^{(n)}, X^{(n-2)})$ and $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$ we have

$$\begin{tikzcd}
 & H_n(X^{(n-1)}, X^{(n-2)}) \arrow[r, equal] \arrow[d] & 0 & 0 \arrow[d, equal] \\
H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r, "\Delta"] \arrow[d, "\partial"'] & H_n(X^{(n)}, X^{(n-2)}) \arrow[r] \arrow[d, "j_*"'] & H_n(X^{(n+1)}, X^{(n-2)}) \arrow[r] & H_n(X^{(n+1)}, X^{(n)}) \\
H_n(X^{(n)} \arrow[ur, "i_*"] & H_n(X^{(n)}, X^{(n-1)}) \arrow[r, equal] \arrow[d, "\partial_D"'] & D_n & \\
 & H_{n-1}(X^{(n-2)}, X^{(n-1)}) \arrow[r, equal] & D_{n-1} &
\end{tikzcd}$$

$j_* i_*$ is induced by the canonical map of pairs $\big(X^{(n)}, \emptyset\big) \to \big(X^{(n)}, X^{(n-1)}\big)$ so $j_*\Delta = j_* i_* \partial = \partial_D$.

The diagram shows that $\ker(\partial_D)_n \cong H_n(X^{(n)}, X^{(n-1)})$.

Therefore $H_n(D_*) = \ker(\partial D)_n / \operatorname{Im}(\partial D)_n \cong H_n(X^{(n)}, X^{(n-1)}) / \operatorname{Im}\Delta \cong H_n(X^{(n+1)}, X^{(n-2)})$.

$$\begin{tikzcd}
H_n(X^{(n-2)}) \arrow[r] \arrow[d, equal] & H_n(X^{(n+1)}) \arrow[r, "\cong"] & H_n(X^{(n+1)}, X^{(n-2)}) \arrow[r] & H_{n-1}(X^{(n-2)}) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

Thus $H_n(D_*) \cong H_n(X^{(n+1)}, X^{(n-2)}) \cong H_n(X^{(n+1)}) \cong H_n(X)$ □

9.7.1 Application: Calculation of $J_*(\mathbb{R}P^n)$ (for $1 \le n \le \infty$)

$p : S^n \to \mathbb{R}P^n$ $\quad$ p = quotient map (covering projection)

Want to find "compatible" CW-complex structures on S^n and $\mathbb{R}P^n$ (i.e. such that p is a "cellular" map).

$S^n = e_0^+ \cup e_0^- \cup e_1^+ \cup e_1^- \cup \ldots \cup e_n^+ \cup e_n^-$ where $e_j^+ = \{(x_0, \ldots, x_j) \in S^j \mid x_j > 0\}$.

Let $e_j = p(e_j^+) \subset \mathbb{R}P^n$.

$p\big|_{e_j^+}$ is a homeomorphism. In fact, $e_j = p(e_j^+) = p(e_j^-)$ is an evenly covered open set in $\mathbb{R}P^n$ with $p^{-1}(e_j) = e_j^+ \cup e_j^-$. So e_j is an open j-cell and $\mathbb{R}P^n = e_0 \cup e_1 \cup \ldots \cup e_n$ is a CW-complex structure on $\mathbb{R}P^n$ (and p is a cellular map).

We define $\mathbb{R}P^\infty := \cup_n \mathbb{R}P^n = e_0 \cup e_1 \cup \ldots \cup e_n \cup \ldots$ and topologize it by declaring that $A \subset \mathbb{R}P^n$ shall be closed if and only if $A \cup \overline{e_n}$ is closed in $\overline{e_n}$ for all n. Thus by construction $\mathbb{R}P^\infty$ is also a CW-complex.

p induces a map of cellular chain complexes $p_* : D_*(S^n) \to D_*(\mathbb{R}P^n)$.

$D_j(S^n) \cong \mathrm{F_{ab}}\{j\text{-cells of} S^n\} \cong \mathbb{Z} \oplus \mathbb{Z}$ $\qquad$ $D_j(\mathbb{R}P^n) \cong \mathbb{Z}$

$$\begin{tikzcd}
& \mathbb{Z}\oplus\mathbb{Z} \arrow[d, equal] & \mathbb{Z}\oplus\mathbb{Z} \arrow[d, equal] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[d, equal] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[d, equal] & \\
0 \arrow[r] \arrow[d] & D_n(S^n) \arrow[r, "\partial"] \arrow[d, "p_*"] & D_{n-1}(S^n) \arrow[r, "\partial"] \arrow[d, "p_*"] & \ldots \arrow[r, "\partial"] & D_j(S^n) \arrow[r, "\partial"] \arrow[d, "p_*"] & \ldots \arrow[r, "\partial"] & D_0(S^n) \arrow[r] \arrow[d, "p_*"] & 0 \\
0 \arrow[r] & D_n(\mathbb{R}P^n) \arrow[r, "\partial"] \arrow[d, equal] & D_{n-1}(\mathbb{R}P^n) \arrow[r, "\partial"] \arrow[d, equal] & \ldots \arrow[r, "\partial"] & D_j(\mathbb{R}P^n) \arrow[r, "\partial"] \arrow[d, equal] & \ldots \arrow[r, "\partial"] & D_0(\mathbb{R}P^n) \arrow[r] \arrow[d, equal] & 0 \\
& \mathbb{Z} & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} &
\end{tikzcd}$$

To determine $\partial : D_j(\mathbb{R}P^n) \to D_{j-1}(\mathbb{R}P^n)$ first determine $\partial : D_j(S^n) \to D_{j-1}(S^{n-1})$.

Let $a : S^n \to S^n$ denote the antipodal map $a(x) = x$.

a respects the cellular structure of S^n: $\quad a(e_j^+) = e_j^- \qquad a(e_j^-) = e_j^+$
so it induces a chain map $a_* : D_*(S^n) \to D_*(S^n)$.

We pick generators for $D_*(S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows.

In the summand $\mathbb{Z} \subset D_0(S^n)$ corresponding to e_0^+ pick one of the two generators and call it f_0^+. Then $a_*(f_0^+)$ will be a generator for the other $\mathbb{Z}$ summand in $D_0(S^n)$ so set ${}_0^- := a_* f_0^+$.

Lemma 9.7.4 *$f_0^+ - f_0^-$ generates* $\operatorname{Im} \partial$.

Proof: a induces the indentity on $H_0(S^n)$ (any self-map of a connected space does), so $[f_0 f^-] = a_*[f_0^-] = a_*[f_0^+] = [f_0^+]$. Hence $[f_0^+] - [f_0^-]$ is the zero homology class so $f_0^+ - f_0^- \in \operatorname{Im} \partial$.

Since $D_*(S^n)$ is a complex whose homology gives $H_*(S^n)$ and we know $H_0(S^n) \cong \mathbb{Z}$, we conclude that $f_0^+ - f_0^-$ generates $\operatorname{Im} \partial$. ☐

Pick a generator of the $\mathbb{Z}$ summand of $D_1(S^n)$ corresponding to e_1^+ and call it f_1^+. So $\partial f_1^+ = m(f_0^+ - f_0^-)$ for some m. Replacing f_1^+ bye $-f_1^+$ if necessary, we may assume that $m \geq 0$. Let $f_1^- = a f_1^+$. Then $\partial f_1^- = m(a f_0^+ - a f_0^-) = m(f_0^- - a^2 f_0^+) = m(f_0^- - f_0^+) - m(f_0^+ - f_0^-)$. Since $\partial\big(D_1(S^n)\big)$ is generated by ∂f_1^+ and ∂f_1^-, the only way it can be generated by $f_0^+ - f_0^-$ is if $m = 1$.

$\partial f_1^+ = f_0^+ - f_0^- \qquad \partial f_1^- = -(f_0^+ - f_0^-)$

Therefore $\ker \partial_1 : D_1(S^n) \to D_0(S^n)$ is generated by $f_1^+ + f_1^-$. But since $H_1(S^n) = 0$, $\ker \partial_1 = \operatorname{Im} \partial_2$.

Pick a generator $f_2^+ \in D_2(S^n)$ corresponding to e_2^+. Then $\partial f_2^+ = m(f_1^+ - f_1^-)$ for some m, and as above we may assume $m \geq 0$. Let $f_2^- = a_* f_2^+$. Then $\partial f_2^- = m(f_1^- + f_1^+)$ and so as above we conclude that $m = 1$.

$\partial f_2^+ = f_0^+ + f_0^- \qquad \partial f_2^- = f_0^+ + f_0^-$ Therefore $\ker \partial_2$ is generated by $f_2^+ - f_2^-$. As above, pick f_3^+ and f_3^- s.t. $f_3^- = a_* f_3^+$, $\partial f_3^+ = f_2^+ - f_2^-$ and $\partial f_2^- = -(f_2^+ - f_2^-)$.

Continuing, get f_j^+ and f_j^- for $j = 0, \ldots, n$ s.t. $f_j^- = a_* f_j^+$ and $\partial f_j^+ = \partial f_j^- = f_{j-1}^+ - f_{j-1}^-$ when j is even, while $\partial f_j^+ = f_{j-1}^- - f_{j-1}^-$ when j is even, $\partial f_j^- = -(f_{j-1}^- - f_{j-1}^-)$ when j is odd.

For each j, $f_j := p_*(f_j) = p_*(f_j^-) \in D_j(\mathbb{R}P^n)$ since $p_* a_* = p_*$.

Therefore $\partial f_j = \begin{cases} f_{j-1} + f_{j-1} = 2 f_{j-1} & j \text{ even;} \\ f_{j-1} + f_{j-1} = 0 & j \text{ odd.} \end{cases}$

$$\begin{tikzcd} D_*(\mathbb{R}P^n) & {} \arrow[r] & \mathbb{Z} \arrow[r] & \ldots \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

n even:

$$H_q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/(2\mathbb{Z}) & q \text{ odd, } q < n \\ 0 & q \text{ even or } q > n \end{cases}$$

n odd:

$$H_q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ \mathbb{Z}/(2\mathbb{Z}) & q \text{ odd, } q < n \\ 0 & q \text{ even or } q > n. \end{cases}$$

$$\text{That is, } H_q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & n \text{ (if } n \text{ odd)} \\ \vdots & \\ 0 & 4 \\ \mathbb{Z}/(2\mathbb{Z}) & 3 \\ 0 & 2 \\ \mathbb{Z}/(2\mathbb{Z}) & 1 \\ \mathbb{Z} & 0. \end{cases}$$

9.7.2 Complex Projective Space

Regard S^{2n+1} as the unit sphere of $\mathbb{C}^{n+1}$.

An action $S^1 \times S^{2n+1}$ of S^1 on ${}^{2n+1}$ is given by $\big(\lambda, (z_0, \ldots, z_n)\big) \mapsto (\lambda z_0, \ldots, \lambda z_n)$. Note that $|\lambda z_0|^2 + \ldots + |\lambda z_n|^2 = \lambda|(|z_0|^2 + \ldots + |z_n|^2) = 1 \cdot 1 = 1$ so $(\lambda z_0, \ldots, \lambda z_n) \in S^{2n+1}$.

Define as the orbit space $\mathbb{C}P^n := S^{2n+1}/S^1$.

The inclusions $\mathbb{C}^n \overset{i}{\hookrightarrow} \mathbb{C}^{n+1}$, $(z_0, \ldots, z_{n-1}) \mapsto (z_0, \ldots, z_{n-1}, 0)$ respects the S^1 action so i induces $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$.

Proposition 9.7.5 *$\mathbb{C}P^n$ has a CW-structure: $e^0 \cup e^2 \cup \ldots \cup e^{2n}$*

Proof: Suppose by induction that we have given $\mathbb{C}P^{n-1}$ a CW-structure with one cell in each even degree up to $2n-2$: $\mathbb{C}P^{n-1} = e_0 \cup^2 \cup \ldots \cup e^{2n-1}$.

Let $z = (z_0, \ldots, z_n)$ represent a point in $\mathbb{C}P^n$. Then z lies in $\mathbb{C}P^{n-1}$ if and only if $z_n = 0$. By multiplying by a suitable $\lambda \in S^1$ we may choose to new representative for z in which z_n is real and $z_n \geq 0$. Unless $z_n = 0$, z will have a unique representativve of this form. Writing $z_j = x_j + x_j + iy_j$ (with $y_n = 0$) we have $z = (x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n, 0)$ with $x_n \geq 0$.

Let $E^{2n}_+ = \{(w_0, \ldots, w_{2n}) \in S^{2n} \mid w_{2n} \geq 0\}$. E^{2n}_+ is a $2n$-cell.

Define f^{2n} to be the composite $E^{2n}_+ \hookrightarrow S^{2n} \hookrightarrow S^{2n+1} \xrightarrow{\text{quotient}} \mathbb{C}P^n$. (That is, $(w_0, \ldots, w_{2n}) \mapsto [(w_0 + iw_1, w_2 + iw_3, \ldots, w_{2n-2} + iw_{2n-1}, w_{2n})]$.)

$e^{2n} = \{w_0, \ldots, w_{2k} \in S^{2k} \mid w_{2n} > 0\}$. By the above, the restriction of f_{2n} to e^{2n} is a bijection. It is also an open map (by definition of quotient topology a set map is open if an donly if its inverse image is open and the inverse image of $f^{2n}(U)$ is $\cup_{\lambda \in S^1} \lambda \cdot U$) so it is a homeomorphism. Therefore $\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup e^{2n} = e^0 \cup e^2 \cup \ldots \cup e^{2n}$ is a CW-complex.

(Note: By compactness, the $3rd$ condition is automatic when there are only finitely many cells.) □

Can define a CW-comples $\mathbb{C}P^\infty$ by $\mathbb{C}P^\infty := \cup_n \mathbb{C}p^n = e^0 \cup e^2 \cup \ldots \cup e^{2n} \cup \ldots$ topologized by $A \subset \mathbb{C}P^\infty$ is closed if and only if $A \cap \overline{e^2 n}$ is colosed in $\overline{e_n}$ for all n.

Theorem 9.7.6 $H_q(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & q \text{ even, } q \leq 2n \\ 0 & q \text{ odd, } q > 2n. \end{cases}$

Proof:

$$\begin{tikzcd}
0 \arrow[r] & D_{2n}(\mathbb{C}P^n) \arrow[r] \arrow[d, equal] & D_{2n-1}(\mathbb{C}P^n) \arrow[r] \arrow[d, equal] & D_{2n-1}(\mathbb{C}P^n) \arrow[r] \arrow[d, equal] & \ldots \arrow[r] & D_1(\mathbb{C}P^n) \arrow[r] \arrow[d, equal] & D_0(\mathbb{C}P^n) \arrow[r] \arrow[d, equal] & 0 \\
& \mathbb{Z} & 0 & \mathbb{Z} & & 0 & \mathbb{Z} &
\end{tikzcd}$$

Every $2nd$ group is 0 so the boundary maps are all 0. Therefore $H_*(\mathbb{C}P^n)$ is as stated. □

Remark 9.7.7 *Using the same ideas as above, one can define quaternionic projective space* $\mathbb{H}P^n$ *by* $\mathbb{H}P^n := S^{4n+3}/S^3$ *where we think of* S^3 *as the unit sphere of the quaternions* $\mathbb{H}$ *and* S^{4n+3} *as the unit sphere in* $\mathbb{H}P^{n+1}$ *with quaternionic multiplication as the action. n this case we get that* $\mathbb{H}P^n$ *is a CW-complex of the form* $\mathbb{H}P^n = e^0 \cup e^4 \cup \ldots e^{4n}$. *We can also define* $\mathbb{H}P^\infty = \cup_n \mathbb{H}P^n = e^0 \cup e^4 \cup \ldots e^{4n} \ldots$. *As above we get* $H_q(\mathbb{H}P^n) = \begin{cases} \mathbb{Z} & q \equiv 0(4), q \leq 4n; \\ 0 & q \not\equiv 0(4), \text{ or } q > 4n. \end{cases}$

(Details left as an exercise.)