

6.6 Categories and Functors

Definition 6.6.1 A category \mathbf{C} consists of:

E1) A collection of objects (which need not form a set) known as $\text{Obj}(\mathbf{C})$

E2) For each pair X, Y in $\text{Obj}(\mathbf{C})$, a set (denoted $\mathbf{C}(X, Y)$ or $\text{Hom}_{\mathbf{C}}(X, Y)$) called the morphisms in the category \mathbf{C} from X to Y

E3) For each triple X, Y, Z in $\text{Obj}(\mathbf{C})$, a set function $\circ : \mathbf{C}(X, Y) \times \mathbf{C}(Y, Z) \rightarrow \mathbf{C}(X, Z)$ called composition

E4) For each X in $\text{Obj}(\mathbf{C})$, an element $1_X \in \mathbf{C}(X, X)$ called the identity morphism of X such that:

A1) $\forall f \in \mathbf{C}(X, Y), 1_Y \circ f = f$ and $f \circ 1_X = f$.

A2) $f \in \mathbf{C}(X, Y), g \in \mathbf{C}(Y, Z), h \in \mathbf{C}(Z, W) \Rightarrow h \circ (g \circ f) = (h \circ g) \circ f \in \mathbf{C}(X, W)$

Examples:

	Objects	Morphisms	\circ	id
1.	Sets	Set functions	comp. of functions	identity set map
2.	Groups	Group homomorphisms	"	"
3.	Top. spaces	conts. functions	"	"

4. "Topological pairs"

An object in \mathbf{C} is a pair (X, A) of topological spaces with $A \subset X$.

Morphisms $(X, A) \mapsto (Y, B) = \{ \text{conts. } f : X \rightarrow Y \mid f(A) \subset B \}$

5. X p.o. set. Define \mathbf{C} by $\text{Obj}(\mathbf{C}) = X$.

$$\mathbf{C}(x, y) = \begin{cases} \text{set with one element} & \text{if } x \leq y; \\ \emptyset & \text{if } y \leq x \text{ or } x, y \text{ not comparable} \end{cases}$$

6. \mathbf{C} any category. Define \mathbf{C}^{op} by

$$\text{Obj } \mathbf{C}^{\text{op}} = \text{Obj } \mathbf{C}.$$

$$\mathbf{C}^{\text{op}}(X, Y) = \mathbf{C}(Y, X).$$

$$g \circ_{\mathbf{C}^{\text{op}}} f = f \circ_{\mathbf{C}} g.$$

Definition 6.6.2 A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ consists of:

E1) For each object \mathbf{X} in \mathbf{C} , an object $F(\mathbf{X})$ in \mathbf{D}

E2) For each morphism g in $\mathbf{C}(X, Y)$, a morphism $F(g)$ in $\mathbf{D}(F(\mathbf{X}), F(\mathbf{G}))$

such that:

A1) $F(1_{\mathbf{X}}) = 1_{F(\mathbf{X})}$

A2) $F(g \circ f) = F(g) \circ F(f)$

Examples:

1. “Forgetful” functor $F : \text{Top Spaces} \rightarrow \text{Sets}$

$F(X) =$ underlying set of top. space X

2. Sets \rightarrow \mathbf{k} -vector spaces

$S \mapsto$ “Free” vector space over \mathbf{k} on basis S

$(S \rightarrow TF(S)) \mapsto F(T)$

3. Completely regular topological spaces and continuous maps \rightarrow Compact topological spaces and conts. maps

$X \mapsto \beta(X)$

4. Top. spaces \rightarrow Compactly generated top. spaces

$X \mapsto X_{\mathbf{k}}$

Definition 6.6.3 If F and G are functors from \mathbf{C} to \mathbf{D} , then a natural transformation $n : F \rightarrow G$ consists of:

For all X in \mathbf{C} , a morphism $n_X \in D(F(X), G(X))$ s.t. $\forall f \in \mathbf{C}(X, Y)$,

$$\begin{array}{ccc} F(X) & \xrightarrow{n_X} & G(X) \\ \downarrow F(f) & & \downarrow G(f) \\ F(Y) & \xrightarrow{n_X} & G(Y) \end{array}$$

commutes.

Example: \mathbf{C} = topological pairs

\mathbf{D} = topological spaces

$F : \mathbf{C} \rightarrow \mathbf{D}$ forget A . i.e. $(X, A) \mapsto X$

$G : \mathbf{C} \rightarrow \mathbf{D}$ $(X, A) \mapsto X/A$

$((X, A) \xrightarrow{f} (Y, B)) \mapsto (X/A \xrightarrow{G(f)} Y/B)$.

$n : F \rightarrow G$ by $n_X : F(X, A) \rightarrow G(X, A)$ is the canonical projection, $X \rightarrow X/A$.

Then $(X, A) \xrightarrow{f} (Y, B)$ yields

$$\begin{array}{ccc} X & \xrightarrow{F(f)} & Y \\ \downarrow n_X & & \downarrow n_Y \\ X/A & \xrightarrow{G(f)} & Y/B. \end{array}$$