9.5 Applications of Homology

First we need some calculations.

Theorem 9.5.1 *Suppose $n > 0$. Then* $H_q(S^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & \text{otherwise.} \end{cases}$

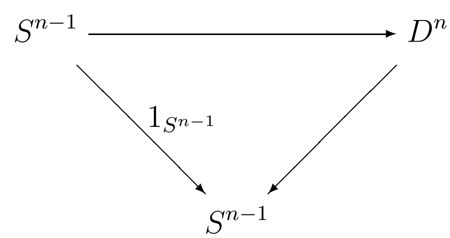
Proof: By induction on n using Mayer-Vietoris. □

Corollary 9.5.2 *S^n is not homotopy equivalent (and in particular not homeomorphic) to S^m for $n \neq m$.* □

Corollary 9.5.3 *$\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for $n \neq m$.*

Proof: If $\mathbb{R}^n$ were homotopy equivalent to $\mathbb{R}^m$ then $\mathbb{R}^n \setminus \{*\}$ would be homeomorphic to $\mathbb{R}^m \setminus \{*\}$. But $S^{n-1} \simeq \mathbb{R}^n \setminus \{*\}$ and $S^{m-1} \simeq \mathbb{R}^m \setminus \{*\}$. □

Theorem 9.5.4 *$\nexists f : D^n \to S^{n-1}$ s.t.*



commutes. □

Corollary 9.5.5 *(Brouwer Fixed Point Theorem) Let $g : D^n \to D^n$. Then $\exists x \in D^n$ s.t. $g(x) = x$.*

Proof: Same as proof in case $n = 2$. □

Definition and Notation:

Let X be a topological space. Define the (unreduced) *cone* on X, denoted CX by $CX := \frac{X \times I}{X \times \{0\}}$.

CX is contractible $\forall X$. ($H : CX \times I \to CX$ by $H\big((x,s),t\big) := (x, st)$.)

Define the (unreduced) suspension of X, denoted SX, by $SX := \frac{X \times I}{X \times \{0\} \cup X \times \{1\}}$. SS^n is homeomorphic to S^{n+1}

C and S are functors from Topological Spaces to Topological Spaces. e.g. Given $f : X \to Y$, $\exists$ induced $S(f) : SX \to SY$ given by $(x,t) \mapsto \big(f(x), t\big)$ satisfying $S(1) = 1$ and $S(g \circ f) = S(g) \circ S(f)$.

Theorem 9.5.6 *(Suspension) $\exists$ a natural isomorphism $\tilde{H}_q(X) \cong \tilde{H}_{q+1}(SX)$ $\forall q$ and $\forall X$.*

Note: Natural means, $\forall f : X \to Y$,
$$\begin{array}{ccc} \tilde{H}_q(X) & \xrightarrow{\cong} & \tilde{H}_{q+1}(SX) \\ \downarrow f_* & & \downarrow Sf_* \\ \tilde{H}_q(X) & \xrightarrow{\cong} & \tilde{H}_{q+1}(SX) \end{array}$$
commutes.

Proof: Let C^+X and C^-X denote the upper and lower cones on X, within SX. Enlarge them slightly to open sets. i.e. Replace them by

$$C^+X := \frac{X \times (\frac{1}{2} - \epsilon, 1)}{X \times \{1\}}, \qquad C^-X := \frac{X \times (0, \frac{1}{2} + \epsilon)}{X \times \{0\}}.$$

Then we have Mayer-Vietoris sequences for C^+X, C^-X, where $C^+X \cup C^-X = SX$ and $C^+X \cap C^-X \simeq X$

$$\begin{array}{ccccccccc} 0 & & & & & & 0 \\ \| & & & & & & \| \\ \tilde{H}_{q+1}(C^+X) \oplus \tilde{H}_{q+1}(C^-X) & \longrightarrow & \tilde{H}_{q+1}(SX) & \xrightarrow[\cong]{\Delta} & \tilde{H}_q(X) & \longrightarrow & \tilde{H}_q(C^+X) \oplus \tilde{H}_{q+1}(C^-X) \\ \downarrow & & \downarrow (Sf)_* & & \downarrow f_* & & \downarrow \\ \tilde{H}_{q+1}(C^+Y) \oplus \tilde{H}_{q+1}(C^-Y) & \longrightarrow & \tilde{H}_{q+1}(SY) & \xrightarrow[\cong]{\Delta} & \tilde{H}_q(Y) & \longrightarrow & \tilde{H}_q(C^+Y) \oplus \tilde{H}_{q+1}(C^-Y) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

$\square$

Remark 9.5.7 *Under the presence of the other axioms, Suspension$\Leftrightarrow$Mayer-Vietoris$\Leftrightarrow$Excision.*

Theorem 9.5.8 *Let $f : S^n \to S^n$ be the reflection $(x_0, \ldots, x_n) \mapsto (-x_0, \ldots, x_n)$. Then $r_* : \mathbb{Z} \cong \tilde{H}_n(S^n) \to \tilde{H}_n(S^n) \cong \mathbb{Z}$ is multiplication by -1.*

Proof: Notice that if we denote $r : S^n \to S^n$ by r_n then $r_n = Sr_{n-1}$. Therefore by naturality of suspension it suffices to prove the theorem in the case $n = 0$ when it is trivial. $\square$

Corollary 9.5.9 *Let : $S^n \to S^n$ be the antipodal map $x \mapsto -x$. Then $a_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ is multiplication by $(-1)^{n+1}$.*

Proof: : Write a as the composition of the $n+1$ reflections $r_j : S^n \to S^n$ given by $r_j(x_0, \ldots, x_n) := (x_0, \ldots, -x_j, \ldots, x_n)$. □

Definition 9.5.10 *Let $f : S^n \to S^n$. Then $f_* : \mathbb{Z} \cong \tilde{H}_n(S^n) \to \tilde{H}_n(S^n) \cong \mathbb{Z}$ is multiplication by k for some integer k. k is called the* degree *of f.*

Theorem 9.5.11 *Let $f : S^n \to S^n$. Suppose $\deg f \neq (-1)^{n+1}$. Then f has a fixed point.*

Proof: If f has no fixed point then the great circle joining $f(x)$ to $-x$ has a well defined shorter and longer segment. Contruct a homotopy $H : f \simeq a$ by moving $f(x)$ towards $-x$ along the shorter seqment. Explicitly, $H(x,t) = \frac{(1-t)f(x)+t(-x)}{||(1-t)f(x)+t(-x)||}$. (The only way the denominator can be zero is if $(1-t)f(x) = tx$ which is doesn't hold for $t = 0$ or 1 and would otherwise require that $f(x) = tx/(1-t)$ which doesn't hold since $f(x)$ is never a multiple of x.) Hence $\deg f = \deg a = (-1)^{n+1}$, which is a contradiction. □

Theorem 9.5.12 *Let $f : S^n \to S^n$. If $\deg f \neq 1$, then $f(x) = -x$ for some x.*

Proof: Since $\deg f \neq 1$, $\deg af \neq (-1)^{n+1}$, so af has a fixed point x. i.e. $x = af(x) = -f(x)$. Hence $f(x) = -x$. □

Theorem 9.5.13 *$\exists$ continuous nowhere vanishing "vector field" on S^n if and only if n is odd. That is, if $T(S^n)$ denotes the tangent bundle to S^n then ($\exists$ continuous $v : S^n \to T(S^n)$ s.t. $v(x) \neq 0$ $\forall x \in S^n$) if and only if n is odd.*

Proof:
$\longleftarrow$ If n is odd, then $v(x_0, x_1, \ldots, x_{2n+1}) := (-x_1, x_0, \ldots, -x_{2n+1}, x_{2n}$ is a nowhere vanishing vector field on S^n.
$\longrightarrow$ Suppose $\exists$ such a v. Define $w : S^n \to S^n$ by $w(x) := v(x)/||v(x)\}||$. Then $x \perp w(x)$ $\forall x \in S^n$. In particular, $w(x) \neq x$ $\forall x$ and $w(x) \neq -x$ $\forall x$. Thus w has no fixed point and hence $\deg w = (-1)^{n+1}$. But since $\nexists x$ s.t. $w(x) = -x$ we also have $\deg w = 1$. Hence $1 = (-1)^{n+1}$, so n is odd.

An alternate more direct argument (not using the two preceding theorems) is as follows:

To get the conclusion $1 = (-1)^{n+1}$ is suffices to show that both $w \simeq 1_{S^n}$ and $w \simeq a$ hold. Define $F : S^n \times I \to S^n$ by $F(x,t) := x\cos(t\pi) + w(x)\sin(t\pi)$. Then $F_0 = 1$, $F_{1/2} = w$ and $F_1 = a$ so F provides a homotopy from 1 to a. Therefore by the homotopy axiom $1 = (-1)^{n+1}$. □