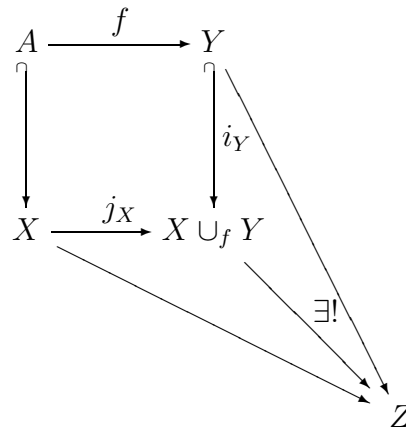


## 6.2 Attaching Maps

Given  $A \subset X$  with  $f : A \rightarrow Y$ , we define “the space obtained from  $Y$  by attaching  $X$  by means of  $f$ ” (written  $X \cup_f Y$ ) as

$$X \cup_f Y = (X \amalg Y) / \sim$$

where  $a \sim f(a) \forall a \in A$ .



is a pushout in the category of topological spaces.

$i_Y$  is always an injection.

$j_X$  is an injection iff  $f$  is.

**Example 6.2.1**  $Y = *$   $f : A \rightarrow *$ .

Then  $X \cup_f * = X/A$ .

Associativity:  $A \subset X$ ,  $B \subset Y$ .

$f : A \rightarrow Y$ ,  $g : B \rightarrow Z$ .

Then

$$X \cup_{j_Y \circ f} (Y \cup_g Z) \cong (X \cup_f Y) \cup_g Z = \frac{X \amalg Y \amalg Z}{\sim}$$

Assume  $A$  is closed in  $X$ .

**Proposition 6.2.2**

1. In  $X \cup_f Y$ ,  $i_Y(Y)$  is closed,  $j_X(X \setminus A)$  is open.
2. (a)  $i_Y : Y \cong i_Y(Y)$ ,  
(b)  $j_X : X \setminus A \cong j_X(X \setminus A)$ .

**Proof:**

1.  $X \cup_f Y = i_Y(Y) \cup j_X(X \setminus A)$  and  $i_Y(Y) \cap j_X(X \setminus A) = \emptyset$   
 $\pi : X \amalg Y \rightarrow X \cup_f Y$   
 $\pi^{-1}(j_X(X \setminus A)) = X \setminus A$  open in  $X \amalg Y$   
Therefore  $j_X(X \setminus A)$  open in  $X \cup_f Y$   
Therefore  $i_Y(Y)$  closed
  2. (a) Show  $Y$  open in  $Y \Rightarrow i_Y(U)$  open in  $i_Y(Y)$   
Notice that  $i_Y(Y) = A \cup_f Y \subset X \cup_f Y$   
 $\pi^{-1}(i_Y(U)) = f^{-1}(U) \amalg U$  open in  $A \amalg Y$ .  
Therefore  $i_Y(U)$  open in  $A \cup_f Y = i(Y)$
  - (b) Show  $V$  open in  $X \setminus A \Rightarrow j_X(V)$  open in  $j_X(X)$   
 $\pi^{-1}(j_X(V)) = V$  open in  $A \amalg Y$   
Therefore  $j_X(V)$  open in  $X \cup_f Y$   
Therefore  $j_X(V)$  open in  $j_X(X)$  (since it is even open in entire space)
- 

From now on we think of  $Y$  as the subset  $i_Y(Y)$  of  $X \cup_f Y$ .

**Corollary 6.2.3**  $F \subset X \cup_f Y$  is closed  $\Leftrightarrow F \cap i_Y(Y)$  and  $F \cap \overline{j_X(X \setminus A)}$  are closed.

**Proof:** Since  $X \cup_f Y = i_Y(Y) \cup \overline{j_X(X \setminus A)}$  this follows from the fact that  $i_Y(Y)$  is closed. □

**Proposition 6.2.4** If  $X$  and  $Y$  are compact, then  $X \cup_f Y$  is compact.

**Proof:**  $X, Y$  compact  $\Rightarrow X \amalg Y$  compact  $\Rightarrow X \cup_f Y = \pi(X \amalg Y)$  compact □

**Proposition 6.2.5** If  $X$  and  $Y$  are normal, then  $X \cup_f Y$  is also normal.

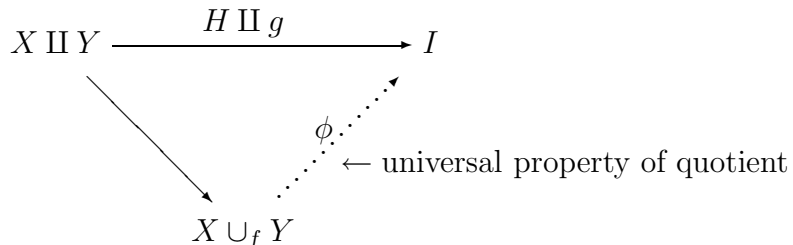
**Proof:** Suppose  $B, C \subset X \cup_f Y$  with  $B \cap C = \emptyset$  where  $B, C$  are either closed or singletons. (We don't assume singletons are closed — have to show  $T_1$  as well)

Then  $B \cap Y, C \cap Y$  are disjoint closed subsets of  $Y$  so  $\exists g : Y \rightarrow I$  s.t.  $g(B \cap Y) = 0, g(C \cap Y) = 1$ .

Define  $h : j_X^{-1}(B) \cup j_X^{-1}(C) \cup A \rightarrow I$  by  $h|_{j_X^{-1}(B)} = 0, h|_{j_X^{-1}(C)} = 1, h_A = g \circ f$ .

This agrees on overlaps (which are closed) so yields a well-defined cont. function.

Domain of  $h$  closed in  $X, X$  normal  $\xrightarrow{\text{(Tietze)}} \exists H : X \rightarrow I$  extending  $h$ .



$$\phi(B) = 0, \phi(C) = 1,$$

Therefore  $\exists$  open sets separating  $B$  and  $C$ . Applied to singletons gives Hausdorff (thus  $T_1$ ) and then applied again to closed sets gives normal.  $\square$

**Proposition 6.2.6** *If  $Y$  is Hausdorff and  $X$  is metric, then  $X \cup_f Y$  is Hausdorff.*

**Proof:**

$$1. x \neq w \in X \setminus A$$

Separation in  $X \setminus A$  gives a separation in  $X \cup_f A$  since  $X \setminus A$  is open.

$$2. X \in X \setminus A, y \in Y$$

Find  $\epsilon > 0$  s.t.  $N_{2\epsilon}(x) \subset X \setminus A$

Then  $x \in N_\epsilon(x) \subset \overline{N_\epsilon(x)} \subset X \setminus A$ , (where the closure can be taken either in  $X \setminus A$  or in  $X \cup_f Y$  — it's the same)

Then  $N_\epsilon(x), \left(\overline{N_\epsilon(x)}\right)^c$  separate  $x$  and  $y$ .

$$3. y_1, y_2 \in Y$$

**Lemma 6.2.7**  *$X$  metric.  $A \subset X$ .  $V$  open in  $A$ .*

*Then  $\exists$  open  $U$  in  $X$  s.t.  $U \cap A = V$  and  $\overline{U} \cap A = \text{closure of } V \text{ in } A$*

**Proof:**

See Problem Set I.  $\square$

**Proof of Prop. (cont):**

Let  $U', V'$  be a separation of  $y_1, y_2$  in  $Y$  (with  $y_1 \in U', y_2 \in V'$ )

$f^{-1}(U')$  open in  $A$ .  $X$  metric so by Lemma,  $\exists$  open  $U$  in  $X$  s.t.  $U \cap A = f^{-1}(U')$ ,  $\overline{U} \cap A = \text{closure of } f^{-1}(U') \text{ in } A = \overline{f^{-1}(U')}$  (since  $A$  closed).

Let  $W = (j_X U) \cup U' \subset X \cup_f Y$

$\pi^{-1}(\text{any}) = j_X^{-1}(\text{any}) \amalg i_Y^{-1}(\text{any})$

Since  $j_X^{-1}(U' \cup j_X(U)) = f^{-1}(U') \cup U = U$  and  $i_Y^{-1}(U' \cup j_X(U)) = U' \cup (j_X(U) \cap Y) = U' \cup f(U \cap A) = U'$  we get  $\pi^{-1}(W) = U \amalg U'$  in  $X \amalg Y$  so  $W$  is open in  $X \cup_f Y$ .

**Claim:**  $\overline{W} = j_X(\overline{U}) \cup \overline{U'}$

**Proof:**  $B \subset f^{-1}(f(B)) \Rightarrow \overline{B} \subset f^{-1}(f(\overline{B})) \Rightarrow f(\overline{B}) \subset \overline{f(B)}$  in general, and so  $W \subset \overline{U'} \cup j_X(\overline{U}) \subset \overline{U'} \cup \overline{j_X(U)} = \overline{W}$ .

Therefore sufficient to show that  $\overline{U'} \cup j_X(\overline{U})$  is closed.

**SubClaim:**  $j_X^{-1}(j_X(\overline{U} \cup \overline{U'})) = \overline{U} \cup j_X^{-1}(\overline{U'})$

**Proof:**  $\overline{U} \subset j_X^{-1}j_X(\overline{U})$  so RHS  $\subset$  LHS.

Conversely, suppose that  $a \in$  LHS.

If  $a \in j_X^{-1}(\overline{U'})$  then  $a \in \text{RHS}$  and if  $a \in \overline{U}$  then  $a \in \text{RHS}$ .

So suppose  $a \in (j_X^{-1}j_X\overline{U}) \setminus \overline{U}$ .

Then  $\exists b \in \overline{U}$  s.t.  $j_X(a) = j_X(b)$ . Since  $a \neq b$  this implies  $a, b \in A$ . Hence  $b \in \overline{U} \cap A = \text{closure of } f^{-1}(U') \text{ in } A$ .

If  $Z$  is a nbhd. of  $j_X(b)$  then  $j_X^{-1}(Z)$  is a nbhd. of  $b$ , so  $j_X^{-1}(Z)$  contains pts. of  $V$ . Hence  $Z$  contains pts. of  $j_X(f^{-1}(U')) \subset U'$ . True  $\forall$  nbhds. of  $j_X(b)$ , so  $j_X(a) = j_X(b) \in \overline{U'}$ .

Therefore  $a \in j_X^{-1}(\overline{U'}) \in \text{RHS}$ .

**Proof of Claim (cont.):**

SubClaim  $\Rightarrow j_X^{-1}(j_X(\overline{U}) \cup \overline{U'}) = \overline{U} \cup j_X^{-1}(\overline{U'})$  closed in  $X$

$$i_Y^{-1}(j_X(\overline{U}) \cup \overline{U'}) = (j_X(\overline{U}) \cap Y) \cup (\overline{U'} \cap Y) \quad (6.1)$$

$j_X(\overline{U}) \cap Y = j_X(\overline{U} \cap A) = f(\overline{U} \cap A) = f(\text{closure of } f^{-1}(U') \text{ in } A) \subset \text{closure of } f(f^{-1}(U'))$  in  $Y \subset \overline{U'} \cap Y$ , and so (6.1)  $\Rightarrow i_Y^{-1}(j_X(\overline{U}) \cup \overline{U'}) = \overline{U'} \cap Y$  which is closed in  $Y$ .

Therefore we have shown that  $\pi^{-1}(j_X(\overline{U}) \cup \overline{U'}) = \text{closed} \amalg \text{closed}$  so  $j_X(\overline{U} \cup \overline{U'})$  closed, as desired.

**Proof of Prop. (cont.):**

$y_1 \in U' \subset W$

Show  $y_2 \notin \overline{W}$  so that  $W, (\overline{W})^c$  is the desired separation.

Suppose  $y_2 \in \overline{W}$ . Then  $y_2 \in \overline{W} \cap Y = i_Y^{-1}(\overline{W}) \subset \overline{U'} \cap Y = \text{closure of } U' \text{ in } Y$ .

But  $y_2 \in V'$  and  $V' \cap (\text{closure of } U' \text{ in } Y) = \emptyset$

$\Rightarrow \Leftarrow$

So  $y_2 \notin \overline{W}$ , as desired. □

## 6.3 Coherent Topologies

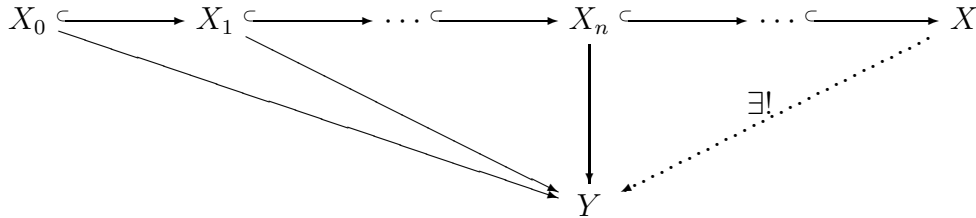
Let  $X_1 \subset X_2 \subset \cdots \subset X_n \subset$  be topological spaces.

Let  $X = \cup_n X_n$

The *coherent topology* on  $X$  defined by the subspaces  $X_n$  is the topology whose closed sets are  $\{A \subset X \mid A \cap X_n \text{ is closed in } X_n \forall n\}$ . (Clearly this collection is closed under intersections and finite unions.) This is the weakest topology on  $X$  s.t. all the inclusion maps are continuous.

Notation: Write  $X = \varinjlim_n X_n$  for  $\cup_n X_n$  with this topology.

**Proposition 6.3.1** Given  $f_n : X_n \rightarrow Y$  s.t.  $f_n|_{X_k} = f_k$  for  $k < n$ ,  $\exists! f : X \rightarrow Y$  s.t.  $f|_{X_n} = f_n$ .



**Proof:** Let  $f$  be the unique set map on  $X$  restricting to  $f_n$  on  $X_n$ . Given closed  $A$  in  $Y$ ,  $f^{-1}(A) \cap X_n = f_n^{-1}(A)$  which is closed in  $X_n$ . Hence  $f^{-1}(A)$  is closed in  $X$ . Therefore  $f$  is continuous.  $\square$

**Proposition 6.3.2** Suppose  $\forall n$  that  $X_n$  is normal and  $X_n$  is closed in  $X$ . Then  $X$  is normal.

**Proof:**

$\forall x \in X, \{x\} \cap X_n = \{\{x\} \text{ or } \emptyset\} = \text{closed in } X_n$

Hence  $\{x\}$  closed.

So  $X$  is  $T_1$ .

Suppose  $A, B$  closed in  $X$  with  $A \cap B = \emptyset$ .

$X_1$  normal  $\Rightarrow \exists g_1 : X_1 \rightarrow I$  s.t.  $g_1(X_1 \cap A) = 0, g_1(X_1 \cap B) = 1$ .

Suppose  $g_n : X_n \rightarrow I$  has been defined s.t.  $g_n(X_n \cap A) = 0, g_n(X_n \cap B) = 1$ ,

$g_n|_{X_k} = g_k$  for  $k < n$ .

To define  $g_{n+1}$ :

Define  $f_n : Y_n := X_n \cup A \cup B \rightarrow I$  by  $f_n(X_n) = g_n, f_n(A) = 0$ , and  $f_n(B) = 1$ .

$A, B, X_n$  closed and  $f_n$  agrees on the overlaps, so  $f_n$  is continuous.

$Y_n$  closed in  $X \Rightarrow Y_n \cap X_{n+1}$  closed in  $X_{n+1}$ , so by Tietze (using  $X_{n+1}$  normal)

$\exists g_{n+1} : X_{n+1} \rightarrow I$  extending  $f_n|_{Y_n \cap X_{n+1}}$ .

Hence  $g_{n+1}|_{X_n} = f_n|_{X_n} = g_n, g_{n+1}(X_{n+1} \cap A) = 0, g_{n+1}(X_{n+1} \cap B) = 1$ .

By universal property of  $\varinjlim, \exists! g : X \rightarrow I$  extending  $g_n \forall n$ .

Then  $g(A) = 0$  and  $g(B) = 1$ .  $\square$

## 6.4 CW complexes

Motivation: Finite CW complexes:

A finite 0-dimensional CW complex consists of a finite set with the discrete topology.

A finite  $(n+1)$ -dimensional CW complex is a space of the form  $(\coprod_{\alpha \in J} D^{n+1}) \cup_f X$  where

- (1)  $X$  is a finite  $k$ -dimensional CW complex for some  $k \leq n$
- (2)  $D^{n+1}$  denotes  $[0, 1]^{n+1}$ .  $\coprod_{\alpha \in J} D^{n+1}$  has the “disjoint union” topology:  $U$  is open if its intersection with each  $D^{n+1}$  is open.
- (3)  $f : \coprod \partial D^{n+1} \rightarrow X$ , where  $S^n \cong \partial D^{n+1} \subset D^{n+1}$

Examples:

- (1)  $I = [0, 1]$
- (2)  $S^n$  which is homeomorphic to  $D^n \cup_f \text{pt} = D^n / \partial D^n$ .

Definition of CW complex which follows is more general and allows for infinite CW-complexes as well.

Terminology:

- Spaces homeomorphic to  $D^m$  will be called *m-cells*.
- Spaces homeomorphic to the interior of  $D^m$  will be called *open m-cells*.
- $m$  is called the *dimension* of the cell.

**Definition 6.4.1** A CW-structure on a Hausdorff space  $X$  consists of a collection of disjoint open cells  $\{e_\alpha\}_{\alpha \in J}$  and a collection of maps  $f_\alpha : D^m \rightarrow X$  s.t.

1.  $X = \cup_{\alpha \in J} e_\alpha$  (disjoint as a set)
2.  $\forall \alpha :$ 
  - (a)  $f_\alpha(\overset{\circ}{D}^m) \subset e_\alpha$  and  $f_\alpha|_{\overset{\circ}{D}^m} : \overset{\circ}{D}^m \cong e_\alpha$
  - (b)  $f_\alpha(\partial D^m) \subset \{\text{union of finitely many of the cells } e_\alpha \text{ having dimension less than } m\}$
3.  $A \subset X$  is closed  $\Leftrightarrow A \cap \overline{e_\alpha}$  is closed in  $\overline{e_\alpha}$  for all  $\alpha$

A space with a CW-structure is called a CW-complex.

To see that this generalizes the above description:

Suppose  $Y = X \cup \left( \coprod_{\beta \in K} D_\beta^{n+1} \right) \cup_g X$  where  $X = \cup_{\alpha \in J} e_\alpha$  is a CW complex with  $\dim e_\alpha \leq n \forall \alpha$ . Write  $C = \coprod_{\beta \in K} D_\beta^{n+1}$  and  $\partial C = \coprod_{\beta \in K} \partial D_\beta^{n+1}$ .

So  $C \setminus \partial C = \coprod_{\beta \in K} \overset{\circ}{D}_\beta^{n+1}$ .

$$\begin{array}{ccc}
\partial C & \hookrightarrow & C \\
\downarrow & & \downarrow j \\
X & \xhookrightarrow{i} & Y
\end{array}$$

Let  $f_\beta = j|_{D_\beta^{n+1}} : D_\beta^{n+1} \rightarrow X$ . (So  $X$  is a union of cells having dimension  $< n + 1$ .)

Since  $Y = \cup_{\alpha \in J} \overline{e_\alpha} \cup \cup_{\beta \in K} \overline{e_\beta}$  in the case of a finite CW complex (where the sets  $J$  and  $K$  are finite) the third condition is automatic.

Terminology:

$\cup \{e_\alpha \mid \dim e_\alpha \leq n\}$  is called the  $n$ -skeleton of  $X$ , written  $X^{(n)}$ .

The restrictions  $f_\alpha|_{\partial D^m}$  are called the *attaching maps*.

Notice that we can recover  $X$  from knowledge of  $X^{(0)}$  and the attaching maps as follows: Inductively define  $X^{(n+1)}$  by  $X^{(n+1)} = \left( \coprod_{\beta \in K_{n+1}} D_\beta^{n+1} \right) \cup_f X^{(n)}$  where  $K_{n+1} = \{\text{all } (n+1)\text{-cells}\}$ . (Knowledge of a map includes knowledge of its domain so we know the set  $K_{n+1}$ .)

$$X^{(0)} \subset X^{(1)} \subset \dots \subset X^{(n)} \dots$$

Define  $X = \cup_n X^{(n)} = \cup_{\alpha \in J} e_\alpha$  and topologize it by condition 3.

If  $\exists M$  s.t.  $X^{(M)} = X$  then  $X$  is called finite dimensional.

$X$  is called *finite* if it has finitely many cells.

Note: A space can have more than one CW-structure giving the same topology.

e.g.

$$S^2 = e_0 \cup e_2$$

$$S^2 = e_0 \cup e_0 \cup e_1 \cup e_1 \cup e_2 \cup e_2$$

Note: The open  $n$ -cells comprising  $X$  are not necessarily open as subsets of  $X$ . Only the top dimensional open cells are actually open in  $X$ .

**Lemma 6.4.2**  $\overline{e_\alpha} = f_\alpha(D^m)$

**Proof:**  $D^m$  compact  $\Rightarrow f_\alpha(D^m)$  compact  $\Rightarrow f_\alpha(D^m)$  closed as  $X$  is Hausdorff. (In fact  $X$  is normal.)

$$e_\alpha = f_\alpha(\overset{\circ}{D}^m) \subset f_\alpha(D^m) \Rightarrow \overline{e_\alpha} \subset f_\alpha(D^m).$$

Conversely  $f_\alpha^{-1}(\overline{e_\alpha}) = \overline{f_\alpha^{-1}(e_\alpha)} = \overline{\text{Int}(D^m)} = D^m$  so  $f_\alpha(D^m) \subset \overline{e_\alpha}$ .  $\square$

**Corollary 6.4.3**  $\overline{e_\alpha} \subset X^{(m)}$ .

**Proof:**  $\overline{e_\alpha} = f_\alpha(D^m) = f_\alpha(\overset{\circ}{D}^m) \cup f_\alpha(\partial D^m)$  with  $f_\alpha(\overset{\circ}{D}^m) = e_\alpha$  and  $f_\alpha(\partial D^m) \subset X^{(m-1)}$ , so  $\overline{e_\alpha} \subset X^{(m)}$ .  $\square$

**Corollary 6.4.4** For any  $\alpha_0$ ,  $\overline{e_{\alpha_0}} \cap e_\alpha = \emptyset$  for all but finitely many  $\alpha$ .

**Proof:** By definition  $f_{\alpha_0}(\partial D^m) \cap e_\alpha = \emptyset$  for all but finitely many  $\alpha$ .  $\square$

**Theorem 6.4.5** A compact  $C \subset X \Rightarrow C \cap e_\alpha = \emptyset$  for all but finitely many  $\alpha$ .

**Proof:**  $X = \cup_{\alpha \in J} e_\alpha$ . Let  $I = \{\alpha \in J \mid C \cap e_\alpha \neq \emptyset\}$ .

For all  $\alpha \in I$ , choose  $y_\alpha \in C \cap e_\alpha$ . Set  $Y = \{y_\alpha\}_{\alpha \in I}$ .

$\forall \beta, \{\alpha \mid \overline{e_\beta} \cap e_\alpha \neq \emptyset\}$  is finite, so  $\overline{e_\beta} \cap Y$  is finite.

Suppose  $S \subset Y$ .

$\forall \beta \in J, S \cap \overline{e_\beta}$  is finite, thus closed in  $X$ , since  $X$  is  $T_1$ .

Hence  $S$  is closed in  $X$ . (Property 3)

In particular,  $Y$  is closed in  $X$  and every subset of  $Y$  is closed in  $Y$ .

So  $Y$  has the discrete topology.

But  $Y \subset C$ ,  $C$  is compact, and  $Y$  is closed, hence  $Y$  is compact. Therefore  $Y$  is discrete implies  $Y$  is finite. Hence  $I$  is finite.  $\square$

**Corollary 6.4.6** If  $A$  is a compact subset of  $X$ , then  $A \subset X^{(N)}$  for some  $N$ .

**Corollary 6.4.7**  $X$  is compact  $\Leftrightarrow X$  is finite.

**Proof:**  $\Rightarrow$  If  $X$  is compact then  $X$  intersects only finitely many  $e_\alpha$ . But  $X$  intersects all  $e_\alpha$  so  $X$  is finite.

$\Leftarrow$   $X^{(n+1)} = C_{n+1} \cup_f X^{(n)}$  where  $C_{n+1} = \coprod_{\beta \in K_{n+1}} D^{n+1}$ .

If  $X$  is finite, then  $K_{n+1}$  is finite, and so  $C_{n+1}$  is compact, and hence  $X^{(n+1)}$  is compact (by induction).

If  $X$  is finite, then  $X = X^{(N)}$  for some  $N$ .  $\square$

## 6.4.1 Subcomplexes

Let  $X = \cup_{\alpha \in J} e_\alpha$  be a CW complex. Suppose  $J' \subset J$ .

$Y = \cup_{\alpha \in J'} e_\alpha$  is called a *subcomplex* of  $X$  if  $\overline{e_\alpha} \subset Y \forall \alpha \in J'$ .

Example:  $X^{(n)}$  is a subcomplex of  $X \forall n$ .

**Proposition 6.4.8** Let  $Y$  be a subcomplex of  $X$ . Then  $Y$  is closed in  $X$ .

**Proof:** For  $\beta \in J$  show  $Y \cap \overline{e_\beta}$  is closed in  $\overline{e_\beta}$ .

$\{\alpha \in J \mid e_\alpha \cap \overline{e_\beta} \neq \emptyset\}$  is finite, so  $\overline{e_\beta} = e_{\alpha_1} \cup \dots \cup e_{\alpha_k}$ .

The  $e_\alpha$  are disjoint so  $Y \cap e_\alpha = \emptyset$  unless  $e_\alpha \subset Y$ .

Discarding those  $\alpha$  for which  $Y \cap e_\alpha = \emptyset$ , write

$$\begin{aligned} Y \cap \overline{e_\beta} &= ((Y \cap e_{\alpha_1}) \cup \cdots \cup (Y \cap e_{\alpha_r})) \cap \overline{e_\beta} && \text{with } e_{\alpha_1}, \dots, e_{\alpha_r} \subset Y \\ &\subset (Y \cap \overline{e_{\alpha_1}}) \cup \cdots \cup (Y \cap \overline{e_{\alpha_r}}) \cap \overline{e_\beta} \\ &\subset (\overline{e_{\alpha_1}} \cup \cdots \cup \overline{e_{\alpha_r}}) \cap \overline{e_\beta} \\ &\subset Y \cap \overline{e_\beta} \end{aligned}$$

using  $e_{\alpha_j} \subset Y$  and applying the definition of subcomplex.

Hence  $Y \cap \overline{e_\beta} = (\overline{e_{\alpha_1}} \cup \cdots \cup \overline{e_{\alpha_r}}) \cap \overline{e_\beta}$  is closed in  $\overline{e_\beta}$ . Hence  $Y$  is closed in  $X$ .  $\square$

**Corollary 6.4.9** *A subcomplex of a CW complex is a CW complex.*

**Proof:** Let  $Y \subset X$  be a subcomplex where  $Y = \bigcup_{\alpha \in J'} e_\alpha$  and  $X = \bigcup_{\alpha \in J} e_\alpha$ .

For  $\alpha \in J'$ ,  $f_\alpha(D^m) = \overline{e_\alpha} \subset Y$  (thus it is in finitely many cells of  $Y$  since  $X$  is a CW-complex) so condition (2) is satisfied.

Check condition (3).

Suppose  $A \cap \overline{e_\alpha}$  closed in  $\overline{e_\alpha}$  for all  $\alpha \in J'$ .

Given  $\beta \in J$ , write  $Y \cap \overline{e_\beta} = (\overline{e_{\alpha_1}} \cup \cdots \cup \overline{e_{\alpha_r}}) \cap \overline{e_\beta}$  with  $\alpha_1, \dots, \alpha_r \in J'$  as above.

Then  $A \cap \overline{e_\beta} = ((A \cap \overline{e_{\alpha_1}}) \cup \cdots \cup (A \cap \overline{e_{\alpha_r}})) \cap \overline{e_\beta}$ .

$A \cap \overline{e_{\alpha_j}}$  is closed in  $\overline{e_{\alpha_j}}$ , thus compact, for  $j = 1, \dots, r$ .

Therefore  $A \cap \overline{e_\beta} = (\text{compact}) \cap \overline{e_\beta} = \text{closed subset of } \overline{e_\beta}$ .

Hence  $A$  is closed in  $X$  and thus closed in  $Y$ .  $\square$

**Corollary 6.4.10**  $X^{(n)}$  is closed in  $X \forall n$ .

**Corollary 6.4.11**  $X = \varinjlim_n X^{(n)}$ .

**Proof:**  $X^{(n)}$  is closed in  $X$  for all  $n$ . If  $A \subset X$  satisfies  $A \cap X^{(n)}$  closed for all  $n$ , then  $\forall \alpha, (A \cap X^{(n)}) \cap \overline{e_\beta} = A \cap \overline{e_\beta}$  closed, since  $\overline{e_\alpha} \subset X^{(n)}$  for some  $n$ .  $\square$

**Proposition 6.4.12**  $X^{(m)}$  is normal  $\forall m$ .

**Proof:**  $X^{(n+1)} = C_{n+1} \cup_f X^{(n)}$  where  $C_{n+1} = \coprod_\beta D^{n+1}$  is normal. Hence  $X^{(m)}$  is normal  $\forall m$  by induction.  $\square$

**Corollary 6.4.13**  $X$  is normal.

There is a stronger theorem which we won't prove which says

**Theorem 6.4.14 (Mizakawa)**  $X$  is a CW-complex  $\Rightarrow X$  is paracompact.

## 6.4.2 Relative CW-complexes

**Definition 6.4.15** A relative CW-structure  $(X, A)$  consists of a Hausdorff space  $X$ , a subspace  $A$  of  $X$ , a collection of disjoint open cells  $\{e_\alpha\}_{\alpha \in J}$  and maps  $f_\alpha : D^m \rightarrow X$  s.t.

1.  $X = A \cup \bigcup_{\alpha \in J} e_\alpha$
2.  $\forall \alpha$ 
  - (a)  $f(D^m) \subset e_\alpha$  and  $f_\alpha|_{D^m_\circ} \cong e_\alpha$
  - (b)  $f_\alpha(\partial D^m) \subset A \cup \{ \text{union of finitely many of the cells } e_\alpha \text{ having dimension less than } m \}$
3.  $B \subset X$  is closed  $\Leftrightarrow B \cap A$  is closed in  $A$  and  $B \cap (A \cup \overline{e_\alpha})$  is closed in  $A \cup \overline{e_\alpha} \forall \alpha$ .

A pair  $(X, A)$  with a relative CW-structure is called a relative CW-complex.

Define  $X^{(n)} = A \cup \bigcup_{\dim e_\alpha \leq n} e_\alpha$ . By convention, set  $X^{(-1)} = A$ .

**Proposition 6.4.16** Let  $(X, A)$  be a relative CW-complex.

1.  $X = \varinjlim_n X^{(n)}$ .
2.  $A$  is normal  $\Rightarrow X$  is normal.
3.  $X^{(n)}$  is closed in  $X \forall n$ .
4.  $(X/A, *)$  is a relative CW complex.

□

## 6.4.3 Product complexes

Let  $X = \bigcup_{\alpha \in J} e^\alpha$  and  $Y = \bigcup_{\beta \in K} e^\beta$  be CW complexes.

Then  $X \times Y = \bigcup_{(\alpha, \beta) \in J \times K} (e_\alpha \times e_\beta)$ .

Note: If  $e_\alpha$  is an  $m$ -cell and  $e_\beta$  is an  $n$ -cell then  $e_\alpha \times e_\beta$  is an  $(m+n)$ -cell.

Define  $f_{\alpha, \beta}$  by  $D^{m+n} = D^m \times D^n \xrightarrow{f_\alpha \times f_\beta} X \times Y$ .

$D^{\circ m+n} = D^{\circ m} \times D^{\circ n} \xrightarrow{f_\alpha \times f_\beta} X \times Y$  is a homeomorphism from  $D^{\circ m+n}$  to its image.

$$\partial D^{m+n} = (\partial D^m \times D^n) \cup (D^m \times \partial D^n) \hookrightarrow X \times Y$$

$$f_{\alpha, \beta}(\partial D^{m+n}) \subset \left\{ ((m-1)\text{-cells}) \times (n\text{-cells}) \right\} \cup \left\{ (m\text{-cells}) \times ((n-1)\text{-cells}) \right\} = \{(m+n-1)\text{-cells}\}.$$

So  $X \times Y$  will be a CW-complex if condition 3 is satisfied. In general, it will not be satisfied.

## 6.5 Compactly Generated Spaces

In this section, all spaces will be assumed to be Hausdorff.

**Definition 6.5.1** A (Hausdorff) space  $X$  is called compactly generated (or a  $k$ -space) if it satisfies  $A \subset X$  is closed  $\Leftrightarrow A \cap K$  is closed in  $K$  for all compact subspaces  $K$  of  $X$ .

**Examples:**

1. Compact spaces
2. CW-complexes

Given  $X$  we define a space  $X_{\mathbf{k}}$  as follows.

As a set,  $X_{\mathbf{k}} = X$ . Topologize  $X_{\mathbf{k}}$  by: closed sets =  $\{A \subset X_{\mathbf{k}} \mid A \cap K \text{ is closed (in the original topology) in } K \text{ for every } K \subset X \text{ which is compact in the original topology}\}$ .

Note: Since  $X$  is Hausdorff,  $A$  closed in  $K$  is equivalent to  $A$  closed in  $X$ .

$A \subset X$  is closed in the original topology  $\Rightarrow A$  is closed in  $X_{\mathbf{k}}$ .

Hence

**Proposition 6.5.2**  $X_{\mathbf{k}} \xrightarrow{\text{id}} X$  is continuous.

Thus the topology on  $X_{\mathbf{k}}$  is finer. In particular  $X_{\mathbf{k}}$  is Hausdorff.

Clearly  $X$  compact  $\Rightarrow X_{\mathbf{k}} = X$ .

**Proposition 6.5.3**  $f : X \rightarrow Y$  continuous implies that  $f$  is continuous when considered as a map  $X_{\mathbf{k}} \rightarrow Y_{\mathbf{k}}$ .

**Proof:** Suppose  $B \subset Y_{\mathbf{k}}$  is closed. If  $K \subset X$  is compact, then  $f(K)$  is compact, so  $B \cap f(K)$  is closed in  $Y$

This implies  $f^{-1}(B \cap f(K))$  is closed in  $X$ . Hence  $f^{-1}(B \cap f(K)) = f^{-1}(B) \cap f^{-1}(f(K)) \supset f^{-1}(B) \cap K$ . So  $f^{-1}(B) \cap K = f^{-1}(B \cap f(K)) \cap K$  which is closed in  $K$ . Hence  $f^{-1}(B)$  is closed in  $X_{\mathbf{k}}$ .  $\square$

**Proposition 6.5.4** If  $A$  is closed in  $X$ , then  $A_{\mathbf{k}}$  is the subspace topology from the inclusion  $A \hookrightarrow X_{\mathbf{k}}$ .

**Proof:**  $A \hookrightarrow X \Rightarrow A_{\mathbf{k}} \hookrightarrow X_{\mathbf{k}}$  is continuous so the  $A_{\mathbf{k}}$  topology is finer than the subspace topology. Suppose that  $B \subset A_{\mathbf{k}}$  is closed. So for all compact  $K \subset A$ ,  $B \cap K$  is closed in  $K$ . We show that  $B$  is closed in  $X_{\mathbf{k}}$ . Suppose  $L \subset X$  is compact.  $A$  is closed, so  $A \cap L$  is a compact subset of  $A$ . However  $B \cap L = B \cap A \cap L$ , so  $B \cap L$  is closed. Hence  $B$  is closed in  $X_{\mathbf{k}}$ .  $\square$

**Corollary 6.5.5**  $K$  is compact in  $X_{\mathbf{k}} \Leftrightarrow K$  is compact in  $X$ .

**Proof:**  $K$  is compact in  $X_{\mathbf{k}} \Rightarrow \text{id}(K) = K$  is compact in  $X$ .

If  $K$  is compact in  $X$ , then  $K$  is closed in  $X$  which implies that  $K_{\mathbf{k}}$  is the subspace topology as a subset of  $X_{\mathbf{k}}$ . Hence  $K$  is compact when regarded as a subspace of  $X_{\mathbf{k}}$ .  $\square$

**Corollary 6.5.6**  $X_{\mathbf{k}}$  is compactly generated.

**Proof:** Suppose  $A \subset X_{\mathbf{k}}$  is such that  $A \cap K$  is closed for all compact  $K$  of  $X_{\mathbf{k}}$ .  $\{\text{compact subspaces of } X_{\mathbf{k}}\} = \{\text{compact subspaces of } X\}$  so this implies  $A$  is closed in  $X_{\mathbf{k}}$ . Hence  $X_{\mathbf{k}}$  is compactly generated.  $\square$

**Proposition 6.5.7** If  $X$  is compactly generated, then  $X_{\mathbf{k}} = X$ . In particular  $(X_{\mathbf{k}})_{\mathbf{k}} = X_{\mathbf{k}}$ .

**Proof:** If  $A$  is closed in  $X$ , then  $A$  is closed in  $X_{\mathbf{k}}$ . Conversely suppose  $A$  is closed in  $X_{\mathbf{k}}$ . Then  $A \cap K$  is closed  $\forall$  compact  $K$  of  $X$ . Hence  $A$  is closed in  $X$ .  $\square$

**Theorem 6.5.8** Let  $X$  and  $Y$  be CW complexes. Then  $(X \times Y)_{\mathbf{k}}$  is a CW complex.

**Proof:** Write  $X = \cup_{\alpha \in J} e_{\alpha}$ , and  $Y = \cup_{\beta \in K} e_{\beta}$ . So as a set  $Z = X \times Y = \cup_{J \times K} e_{\alpha} \times e_{\beta}$ . Since  $D^{m+n}$  is compact,  $f_{\alpha, \beta}(D^{m+n})$  is compact so its topology as a subspace of  $X$  is the same as that as a subspace of  $X \times Y$ . Hence  $f_{\alpha, \beta}$  is continuous as a map from  $D^{m+n}$  to  $Z$  and  $f_{\alpha, \beta}|_{D^{m+n}}$  is still a homeomorphism to its image in  $Z$ , so property (2) in the definition of CW-complex is satisfied. For property (3): Suppose  $A \cap \overline{e_{\alpha} \times e_{\beta}}$  is closed for all  $\alpha, \beta$ . For any compact  $K$ ,  $\pi_1(K)$  and  $\pi_2(K)$  are compact so  $\pi_1(K) \subset \cup_{j=1, \dots, r} e_{\alpha_j}$ ,  $\pi_2(K) \subset \cup_{k=1, \dots, s} e_{\beta_k}$ .

Hence

$$\begin{aligned} K &\subset \cup_{\substack{j=1, \dots, r \\ k=1, \dots, s}} e_{\alpha_j} \times e_{\beta_k} \\ &\subset \cup_{k=1, \dots, s} \overline{\cup_{j=1, \dots, r} e_{\alpha_j} \times e_{\beta_k}} \end{aligned}$$

Hence

$$\begin{aligned} A \cap K &= A \cap \left( \cup_{k=1, \dots, s} \overline{\cup_{j=1, \dots, r} e_{\alpha_j} \times e_{\beta_k}} \right) \cap K \\ &= \left( \cup_{k=1, \dots, s} A \cap \overline{e_{\alpha_j} \times e_{\beta_k}} \right) \cap K \end{aligned}$$

which is closed. So  $A$  is closed in  $Z$ .  $\square$