UNIVERSITY OF TORONTO

MAT 1300Y **Problem Set II**

Due: Friday, Nov. 17, 2006

1. Let G be a topological group and let H be a subgroup of G such that the subspace topology induced on H from G is discrete. Show that for all $x \in G$ there is an open neighbourhood V_x of x such that $V_x \cap hV_x = \emptyset$ for all $h \neq 1$ in H.

2. (Qualifying Exam, Jan. 1995; #1) Let X be connected and let C be a closed subset such that the boundary of C is a single point. Show that C is connected.

3. Show that $\mathbb{C}P^n/\mathbb{C}P^{n-1}$ is homeomorphic to S^{2n}. Use this to define a CW-structure on $\mathbb{C}P^n$.
   Remark: Observe that $\mathbb{C}P^0$ is a single point, so the special case $n = 1$ is the statement that $\mathbb{C}P^1$ is homeomorphic to S^2.

4. Let X be the comb space, defined by

   $$X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1,\ x = 0 \text{ or } x = 1/n\} \cup \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ y = 0\}.$$

   Let $y_0 = (0, 1)$. Show that $1_X \simeq c_{y_0}(\text{rel } \emptyset)$ but that these maps are not homotopic rel $\{y_0\}$.

5. Let X be the Hawaiian earring defined by

   $$X = \cup_{n=1}^{\infty} S_{1/n}(1/n, 0) \subset \mathbb{R}^2$$

   with the subspace topology. Here $S_r(p)$ means the circle of radius r centred at p. Show that X cannot be given the structure of a CW complex.

6. Show that for $n \geq 2$, any map from $\mathbb{R}P^n \to S^1$ is null homotopic.

7. Let $X \subset \mathbb{R}^3$ be the union of S^2 with a chord joining the north and south poles. Describe the universal covering space of X and compute $\pi_1(X)$.

8. Hatcher (page 80; #14) Show that the torus is a 2-fold covering of the Klein bottle. Compute π_1(Klein bottle).