

MAT1300F: Exercises 1

Due date: Friday, October 15, 2010

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NOTE: If you have questions regarding the exercises, please make an appointment to discuss them: office BA6211, tel. (416) 978-5142.

- (1) A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *homogeneous* of degree m if $f(tx) = t^m f(x)$ for all x .
(a) If f is also differentiable, show that

$$\sum_{i=1}^n x_i \partial_i f(x) = m f(x).$$

Hint: if $g(t) = f(tx)$, find $g'(1)$.

- (b) Use this to show that if f is a homogeneous polynomial of degree $m > 0$ then the set of points $\{x \in \mathbb{R}^n : f(x) = a\}$ is an $(n - 1)$ dimensional submanifold of \mathbb{R}^n provided that $a \neq 0$.
(2) If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable and $f(0) = 0$, show that there exist $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ so that

$$f(x) = \sum_{i=1}^n x_i g_i(x)$$

where $x = (x_1, \dots, x_n)$.

Hint: If $h_x(t) = f(tx)$, then

$$f(x) = \int_0^1 \frac{d}{dt} h_x(t) dt.$$

- (3) Regard an $n \times n$ matrix as a point (a_1, \dots, a_n) in $\mathbb{R}^n \times \dots \times \mathbb{R}^n$, where the a_j are the rows of the matrix. Consider the function $F : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by the determinant of the matrix:

$$F(a_1, \dots, a_n) = \det \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix}.$$

Prove that if $x_i \in \mathbb{R}^n$ then

$$dF_{(a_1, \dots, a_n)}(x_1, \dots, x_n) = \sum_{i=1}^n \det \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ x_i \\ \vdots \\ a_n \end{pmatrix}.$$

- (4) Define chart transformations $\Phi_N : S^2 - \{0, 0, 1\} \rightarrow \mathbb{R}^2$ via stereographic projection from the north pole on the plane through the equator: $\Phi_N = (u, v)$ where

$$u(x, y, z) = x/(1 - z); \quad v(x, y, z) = y/(1 - z)$$

Likewise one may provide chart transformations $\Phi_S : S^2 - \{(0, 0, -1)\} \rightarrow \mathbb{R}^2$ by $\Phi_S = (\hat{u}, \hat{v})$ where

$$\hat{u}(x, y, z) = x/(1 + z); \quad \hat{v}(x, y, z) = y/(1 + z).$$

Show that the chart transformations $\Phi_N \circ \Phi_S^{-1}$ are C^∞ .

- (5) Let $S : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$ denote stereographic projection (where N is the north pole $(0, 0, 1)$ and \mathbb{R}^2 is the plane tangent to S^2 at the south pole $(0, 0, -1)$). Recall that S is a bijection.

Let $A : S^2 \rightarrow S^2$ denote the antipodal map.

For $x \in \mathbb{R}^2$, compute $SA S^{-1}(x)$.

- (6) For the map $f : S^m \times S^n \rightarrow S^{mn+m+n}$ defined by

$$(x_0, \dots, x_m), (y_0, \dots, y_n) \mapsto (x_0 y_0, \dots, x_0 y_n, x_1 y_0, \dots, x_1 y_n, \dots, x_m y_0, \dots, x_m y_n)$$

show that $df_{(p,q)}$ is injective at all $(p, q) \in S^m \times S^n$.

Hint: Write $f((x_0, \dots, x_m), (y_0, \dots, y_n))$ as an $(m+1) \times (n+1)$ matrix whose entry in the i -th row and j -th column is $x_i y_j$. Then if $p = (x_0, \dots, x_m)$ and $q = (y_0, \dots, y_n)$, $(df)_{(p,q)}(\xi, \eta)$ may be written in the same way as an $(m+1) \times (n+1)$ matrix. Consider the equations imposed by the columns and rows of the matrix. Use the fact that $\xi = (\xi_0, \dots, \xi_m)$ is in the tangent space to S^m at p and $\eta = (\eta_0, \dots, \eta_n)$ is in the tangent space to S^n at q .

- (7) Show using the regular value theorem that the following are smooth manifolds, and find their dimensions:

(a) $U(n) = \{A \in M_{n \times n}(\mathbb{C}) : AA^\dagger = I\}$ (here A^\dagger is the complex conjugate of the transpose of A)

(b)

$$SL(n, \mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) : \det A = 1\}$$

- (8) The space of k -frames V_n^k is the space of orthonormal k -tuples (v_1, \dots, v_k) in \mathbb{R}^n . Show that the space V_n^k is a smooth manifold of dimension $nk - k(k+1)/2$.
- (9) For an integer $d \geq 0$ the *Brieskorn manifold* $W^{2n-1}(d)$ is defined as the set of points $(z_0, \dots, z_n) \in \mathbb{C}^{n+1}$, where

$$z_0^d + z_1^2 + \dots + z_n^2 = 0$$

$$z_0 \bar{z}_0 + \dots + z_n \bar{z}_n = 2$$

Show that $W^{2n-1}(d)$ is a $(2n-1)$ -dimensional manifold.

- (10) Show that rotation by an angle $0 \leq \alpha \leq 2\pi$ about the z -axis is a C^∞ diffeomorphism of S^2 . Compute the differential of this map at the north pole and at any point on the equator.