MAT 301 - Solution to some problems in Chapter 8

Chapter 24

6) $|S_4| = 4! = 24 = 2^3 \cdot 3$, hence a Sylow 2-subgroup is of order $2^3 = 8$. Let $\rho = (1234), \phi = (12)(34)$, and $H = \{(1), \rho, \rho^2, \rho^3, \beta, \beta\rho, \beta\rho^2, \beta\rho^3\}$. H is a subgroup of S_4 - check that the inverse of each element is in H, as well as the product of any two. H is a Sylow 2-group as it has order 8. Finally, H is isomorphic to D_4. The isomorphism maps (1) to the identity transformation of the square, ρ to an anticlockwise rotation by 90 degrees, and other elements are mapped to what is obtained in the obvious way from labelling the vertices of the square $1, 2, 3, 4$ and matching the permutation with the places of these numbers after the symmetry transformation. See Example 3 in Chapter 5.

8) $|A_4| = 4!/2 = 12 = 2^2 \cdot 3$, so a Sylow 3-subgroup has order 3. In A_4 there are the following cyclic subgroups of order 3: $< (123) >, < (124) >, < (134) >, < (234) >$ (for computing the order, see Theorem 5.3). By the third Sylow Theorem, the number of such groups must be those divisors of 12 which are also $1 \mod 3$. Among the divisors 1,3,4,12, only 1 and 4 equal $1 \mod 3$. Since we found 4 subgroups, the number of them cannot be 1, hence it must be 4, and the above subgroups, being 4 in number, are *all* the Sylow 3-subgroups.

16) $|S_5| = 5! = 120 = 2^3 \cdot 3 \cdot 5$, hence a Sylow 5-subgroup has order 5. Now the divisors of 120 are $1, 2, 2^2 = 4, 2^3 = 8, 3, 5, 2 \cdot 3 = 6, 2 \cdot 5 = 10, 3 \cdot 5 = 15, 2^2 \cdot 3 = 12, 2^2 \cdot 5 = 20, 2^3 \cdot 3 = 24, 2^3 \cdot 5 = 40, 2 \cdot 3 \cdot 5 = 30, 2^2 \cdot 3 \cdot 5 = 60, 2^3 \cdot 3 \cdot 5 = 120$. Of these, only 1 and 6 equal $1 \mod 5$. Hence by the Third Sylow Theorem only these two may be the number of Sylow 5-subgroups. But exhibiting 2 such subgroups will show that their number cannot be 1, hence their number must be 6. Two subgroups of order 5 are $< (12345) >$ and $< (21345) >$ (for computing the order, see Theorem 5.3).

Chapter 27

3) The upper right figure in Figure 1.5, page 37 has symmetry group $\mathbb{Z}_5$, and any such symmetric "star" with 5 vertices would constitute an example. The symmetry group rotates such a star to itself by angle $k/5 \cdot 360$ degrees, $k = 0 \ldots 4$.

8) The 8 transformations $(x, y, z) \rightarrow (\pm x, \pm y, \pm z)$ are the symmetries of such a box, if its center is at the origin. All elements have order 2, except the identity transformation. Also compositions of any two commute, so this is an Abelian group. By the Fundamental Theorem 11.1, an Abelian group of order 8 with 7 elements of order 2 must be $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Note that geometrically, the 3 maps with minus signs in exactly two components have determinant one, so are in SO_3, and represent 180 degree rotations, while the 4 maps with another (non-zero) number of minus signs (i.e. 1 or 3) in all components have determinant -1, so are in $O_3 \setminus SO_3$, and represent reflections.

Matrix group (and symmetry) problems

A) We define a map as follows:

$$A \rightarrow \begin{bmatrix} 1 & 0 & & 0 \\ 0 & & & \\ & & A & \\ 0 & & & \end{bmatrix}, \quad A \in SO_2 \tag{1}$$

and

$$A \rightarrow \begin{bmatrix} -1 & 0 & & 0 \\ 0 & & & \\ & & A & \\ 0 & & & \end{bmatrix}, \quad A \in O_2 \setminus SO_2. \tag{2}$$

In both cases the image matrix has determinant one, so the image is in SO_3 and not just in O_3. The map is clearly one-to-one. To check that this is a homomorphism, one needs to check three equations: for (1) and (1), for (2) and (2), and finally for (1) and (2). Doing the last case, for example, if $A \in SO_2$ and $B \in O_2 \setminus SO_2$, then $AB \in O_2 \setminus SO_2$, and so the image has -1 in its upper left entry. But the product of the image matrix in (1) with the image matrix in (2), but with B replacing A in the latter, is also a matrix with a $1 \cdot (-1) = -1$ in its upper right entry, and AB in the two-by-two slot, as needed. The other cases are computed separately.

B) The matrix is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Geometrically it represents a 45 degree rotation in a plane orthogonal to the y axis.

C) Skipping here the drawing, suppose that the axis of rotation goes through the faces labelled 5 and 6, and the four other faces are labelled consecutively $1, 2, 3, 4$. Then $(1234)(5)(6)$ represents a generator of the cyclic subgroup of order 4 of rotations around this axis. Note that this representation of the symmetry group as a subgroup in S_6 is unrelated to the representation used in the past showing that the symmetry group is isomorphic to S_4 (see Example 7 and Theorem 7.4 in Chapter 7).