MAT 301 - Solution to some of homework 5 (problems in Chapter 9)

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4) The matrix
$$h = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
is in H. The matrices
$$a = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad a^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$
are inverses to each other and in $GL(2,\mathbb{R})$. The product aha^{-1} has a $2,1$-entry equal to -1, so not 0. Hence aha^{-1} is not in H. Therefore H is not normal, by Theorem 9.1.

12) $(14+ < 8 >) + (14+ < 8 >) = 28+ < 8 >= 4+ < 8 >$, $(4+ < 8 >) + (14+ < 8 >) = 18+ < 8 >$, $(18+ < 8 >) + (14+ < 8 >) = 32+ < 8 >= 8+ < 8 >=< 8 >$, where the last equation holds since every element of the form $8 + 8k$ is also of the form $8l$, with $k, l \in \mathbb{Z}$, and this remains true if these numbers are taken $\mod 24$. Since $< 8 >$ is the identity in $\mathbb{Z}_{24}/ < 8 >$, we see that the order of $14+ < 8 >$ is 4.

20) The order of $\mathbb{Z} \oplus \mathbb{Z}/ < (2,2) >$ is infinite: for example, the cosets $(k,0)+ < (2,2) >$ and $(l,0)+ < (2,2) >$ are equal if for any $n \in \mathbb{Z}$ there is $m \in \mathbb{Z}$ with $(k + 2n, 2n) = (l + 2m, 2m)$ but this means $m = n$ from the second component, and then $k = l$ for the second component. This means that $k \neq l$ implies the cosets are distinct, so we get different cosets for distinct choices of $k \in \mathbb{Z}$, hence we get an infinite number of distinct cosets. This group is not cyclic since if it was it would be isomorphic to $\mathbb{Z}$ and hence have no generators of finite order (see Theorem 4.1). But
$$((1,1)+ < (2,2) >) + ((1,1)+ < (2,2) >) = (2,2)+ < (2,2) >=< (2,2) >$$
, so $(1,1)+ < (2,2) >$ has order 2, which is finite. Hence the group is not cyclic.

28) As $165 = 3 \cdot 5 \cdot 11$, and, as primes each two of these factors are relatively prime, and the product of any two of these primes is relatively prime to the third, we have by page 185

$$\begin{aligned} U(165) &= U_{15}(165) \times U_{11}(165) = U_{55}(165) \times U_3(165) = U_{33}(165) \times U_5(165) \quad (1) \\ &= U_{15}(165) \times U_{55}(165) \times U_{33}(165). \quad (2) \end{aligned}$$

50) By Theorem 9.4 and Lagrange's Theorem 7.1

$$|\mathrm{Inn}(\mathrm{D}_4)| = |\mathrm{D}_4/\mathcal{Z}(\mathrm{D}_4)| = |\mathrm{D}_4|/|\mathcal{Z}(\mathrm{D}_4)|.$$

To compute the last number use the fact that D_n has order $2n$, and Example 11 in Chapter 3, which shows that $|\mathcal{Z}(\mathrm{D}_4)| = 2$, to get $|\mathrm{Inn}(\mathrm{D}_4)| = 4$.