MAT 301 - Solution to some problems in Chapter 8

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3) We prove $\phi : G \to G \oplus \{e_H\}$ given by $g \to (g, e_H)$ is an isomorphism. It is well-defined, and $\phi(ab, e_H) = (ab, e_H) = (a, e_H)(b, e_H) = \phi(a)\phi(b)$, so it is operation preserving. It is injective, as $(a, e_H) = (b, e_H)$ implies $a = b$. It is onto, since any $(a, e_H) \in G \oplus \{e_H\}$ is the image of $a \in G$. This concludes the proof, and a similar one shows that $\psi : H \to H \oplus \{e_G\}$ given by $h \to (h, e_G)$ is an isomorphism.

7) $\mathbb{Z} \oplus \mathbb{Z}$ is not cyclic. If it was, $\mathbb{Z} \oplus \mathbb{Z} =< (a, b) >$ for some element (a, b). That element could not have $a = 0$, since sums of $(0, b)$ with itself have 0 in the first component, so an element with a non-zero first component could never be generated. Similarly, the generator cannot have $b = 0$. But now to generate $(2a, b)$ one must have some $n \in \mathbb{Z}$ with $(2a, b) = n(a, b)$, i.e. $(2a, b) = (na, nb)$, so $2a = na$ and $b = nb$. The first of the last two equalities gives, since $a \neq 0$, that $n = 2$. Then the second one becomes $b = 2b$. But as $b \neq 0$, this gives $2 = 1$, which does not hold in $\mathbb{Z}$.

11) As $\gcd(3, 5) = 1$, by Theorem 8.2, especially its Corollary 2, $\mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15}$.

17) By Theorem 4.3, every subgroup of a cyclic group is cyclic. So $G \oplus \{e_H\}$ and $H \oplus \{e_G\}$ are cyclic if $G \oplus H$ is. But any group isomorphic to a cyclic group is cyclic, so problem 3 above shows that G and H are also cyclic.

19) If $(a, b, c) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$ is not the identity, then at least one of a, b, c is not 0. Now $|(a, b, c)| = \text{lcm}(|a|, |b|, |c|)$, and the orders of $|a|, |b|, |c|$ are either 1 or 3, and (since at least one of them is not zero), at least one of them has order 3. Hence $|(a, b, c)| = 3$ for $(a, b, c) \neq (0, 0, 0)$.

23) If $(h, h) \in H$, then its inverse, $(h^{-1}, h^{-1}) \in G \oplus G$ is also in H, and then for two elements in H, $(g, g)(h, h) = (gh, gh)$ is also in H (as both components are equal). Hence, by Theorem 3.2, H is a subgroup. If $G = \mathbb{R}$, the set of real numbers, then $G \oplus G$ is the plane $\mathbb{R}^2$, and H is the line $y = x$ in the plane.

29) As 400 has order 2 in $\mathbb{Z}_{800}$, and 50 has order 4 in $\mathbb{Z}_{200}$, the map $k(1,1) \to k(400,50)$, for $k \in \mathbb{Z}$, is one-to-one and operation preserving from $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ into $\mathbb{Z}_{800} \oplus \mathbb{Z}_{200}$. It is hence isomorphic to its image, which is the required subgroup. This image is $\{0,400\} \oplus \{0,50,100,150\}$.

43) Using Theorem 8.3 and its corollary, $U(165) \cong U(3) \oplus U(5) \oplus U(11) \cong U(15) \oplus U(11) \cong U(33) \oplus U(5) \cong U(3) \oplus U(55)$, as in each of these external direct products, the numbers appearing are relatively prime.

53) We use Theorem 8.3 and the prime power relations near the end of page 155. A simple such relation is that if p is prime, $|U(p)| = \phi(p) = p-1$, where ϕ is the Euler function appearing first in Chapter 4. But the others are different known relations. Hence $U(55) \cong U(5) \oplus U(11) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{10}$, while $U(75) \cong U(5^2) \oplus U(3) \cong \mathbb{Z}_{5^2-5} \oplus \mathbb{Z}_2 = \mathbb{Z}_{20} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{10}$. Here the last isomorphism follows from Corollary 2 to Theorem 8.2.