MAT 301 - Solution to some problems in Chapter 11 (Homework 6)

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2) We can just try successive values of n. We can skip prime numbers as they have only one isomorphism type (cyclic groups):
$n = 4 \Rightarrow \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
$n = 6 \Rightarrow \mathbb{Z}_6, \mathbb{Z}_2 \oplus \mathbb{Z}_3$.
$n = 8 \Rightarrow \mathbb{Z}_8, \mathbb{Z}_4 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
We used here Theorem 11.1 and Theorem 8.2 (especially Corollary 2 for the latter). Hence the answer is $n = 8$.

6) $108 = 2^2 3^3$. Any decomposition $3^3 = 3^{2+1} = 3^{1+1+1}$ corresponds to a factor of $\mathbb{Z}_9 \oplus \mathbb{Z}_3$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, and these give at least two subgroups of order 3, taken from each factor, with the identity element put in the other factors. So to get one subgroup exactly of order 3, the $3^3 = 27$ should not be decomposed as $\mathbb{Z}_{27}$ has a unique such subgroup $< 9 >$. Hence the two groups are $\mathbb{Z}_4 \oplus \mathbb{Z}_{27}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{54}$. Clearly the other factor in these cases has no such subgroup, and as 3 is prime, one cannot get a subgroup of this order by combining other nontrivial subgroups of the two factors. Hence these are two Abelian group satisfying the required condition.

12) Corollary 1 to the Fundamental Theorem 11.1 shows that $|G|$ has a subgroup H of order 10. As it is a subgroup of a finite Abelian group, H is finite Abelian. But the unique isomorphism type of this subgroup is, by Theorem 11.1, just $\mathbb{Z}_2 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{10}$, the last isomorphism by Corollary 2 to Theorem 8.2. And the latter is a cyclic group.

20) If G does not have prime power order, then at least two prime powers in the prime factorization of $|G|$ have a common factor k (Theorem 8.2 and corollaries). This factor is of course also a divisor of G. But then both cyclic groups corresponding to the prime powers via Theorem 11.1 have a subgroup G_k of order k, so by choosing this subgroup in one of the two factors and the identity everywhere else (e.g. $G_k \otimes 0 \otimes 0 \otimes \ldots$ one gets a subgroup of order k in G, and since this can be done for both prime powers, one gets two distinct subgroups of order k, contrary to the assumption.

Hence we may assume that G is isomorphic to a group of prime power order. Take an element x in G of maximal order, say p^l. Any element y in G has order $|y|$ which divides $|x| = | < x > |$, so it is $p^r, 0 \leq r \leq l$. But this means that, as $< x >$ is cyclic, it has a subgroup of order $p^r = |y|$. But we assume there is only one such subgroup in all of G, and $< y >$ is such a subgroup. Therefore, $< y >$ must be a subgroup of

$< x >$. But this holds for y in G, which means that all of G is a subgroup of $< x >$, while also $< x >$ is a subgroup of G. Hence $G =< x >$, so G is cyclic.

32) This problem uses the result of Problem 11 in this chapter. Write

$$G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \ldots \oplus \mathbb{Z}_{n_k}$$

with n_{i+1} dividing n_i. In our argument we will identify the left hand side with the right hand side. An element $b = (b_1, \ldots b_k)$ has maximal order $|b| = \text{lcm}(|b_1|, \ldots |b_k|)$. Now each $|b_j|$ divides n_j (Lagrange for cyclic groups), and each n_j divides n_1. So n_1 is a common multiple of the $|b_i|$'s, so $|b| \leq n_1$. But $a = (1, 0, 0 \ldots 0)$ is an element of order $\text{lcm}(|1|, |0|, |0| \ldots |0|) = \text{lcm}(n_1, 1, 1 \ldots 1) = n_1$. Hence a is an element of maximal order, equal to n_1, and any b in G has order $|b|$ which divides $n_1 = |a|$.

36) If two groups G and H are isomorphic, so are their automorphism groups (the map $\psi\phi\psi^{-1}$, where ϕ is an automorphism of G and $\psi : G \to H$ is an isomorphism, gives a corresponding automorphism of H). So by Corollary 2 to Theorem 8.2, Theorem 8.3 and its corollary and Theorem 6.5,

$$\text{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5) \cong \text{Aut}(\mathbb{Z}_{25}) \cong U(25) \cong \mathbb{U}_2 \oplus \mathbb{U}_3 \oplus \mathbb{U}_5 \cong \mathbb{Z}_1 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$