

**MAT 301 - Solution to some problems in Chapter 10**

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- 1)  $\phi : A \rightarrow \det A$  satisfies  $\phi(AB) = \det(AB) = \det(A) \det(B) = \phi(A)\phi(B)$ .
- 7)  $\phi : G \rightarrow H, \sigma : H \rightarrow K$  so  $\sigma\phi : G \rightarrow K$  and since they are homomorphisms
- $$\sigma\phi(g_1g_2) = \sigma(\phi(g_1g_2)) = \sigma(\phi(g_1)\phi(g_2)) = \sigma(\phi(g_1))\sigma(\phi(g_2)) = \sigma\phi(g_1)\sigma\phi(g_2)$$
- for any  $g_1, g_2 \in G$ , so  $\sigma\phi$  is a homomorphism.
- 11) Take  $\phi((x, y)) = (x \bmod a, y \bmod b)$ . As  $\phi((x, y) + (z, w)) = \phi((x + z, y + w)) = ((x + z) \bmod a, (y + w) \bmod b) = (x \bmod a, y \bmod b) + (z \bmod a, w \bmod b) = \phi((x, y)) + \phi((z, w))$  (where the last few additions are in  $\mathbb{Z}_a$  or  $\mathbb{Z}_b$ ),  $\phi$  is operation preserving. Clearly it is onto and well-defined, so it is a homomorphism onto. As  $\text{Ker}\phi = \{(x, y) | (x \bmod a = 0, y \bmod b = 0)\} = \langle a \rangle \oplus \langle b \rangle \cong \langle (a, 0) \times (0, b) \rangle$ , the isomorphism in question follows from the First Isomorphism Theorem.
- 19) Theorem 10.1 - property 4, implies, as  $\text{Ker}\phi \leq \mathbb{Z}_{17}$  that  $\text{Ker}\phi \neq 0$ , since  $\phi$  is not 1-1 (see Theorem 10.2, property 5). Hence, as  $|\text{Ker}\phi|$  divides 17 by Lagrange,  $|\text{Ker}\phi| = 17$ , so  $\text{Ker}\phi = \mathbb{Z}_{17}$ . Hence  $\phi$  is the trivial map, mapping all of  $\mathbb{Z}_{17}$  to the identity.
- 23) By operation preservation and the fact that 1 is a generator for  $\mathbb{Z}_{20}$ , any homomorphism  $\psi$  is completely determined by  $\psi(1)$  (as  $\psi(k) = k\psi(1)$ ). Now  $\psi$  is onto if and only if  $\psi(1)$  generates  $\mathbb{Z}_{10}$ , and the number of such generators is  $\phi(10) = 4$ , where  $\phi$  is the Euler function. So there are 4 homomorphisms onto, but 10 to, as the onto requirement drops and any element  $l$  in  $\mathbb{Z}_{10}$  can be the image of 1 under  $\psi$ . Note that as  $|l|$  divides 10 it divides 20, so prescribing the homomorphism as above is well-defined.
- 29)  $\phi^{-1}(7) = 7\text{Ker}\phi = 7\{1, 11\} = \{7, 77 \bmod 30\} = \{7, 17\}$ , using property 5 of Theorem 10.1.
- 37) Define  $\phi : K \rightarrow KN/N$  via  $k \rightarrow kN$ . Note that as  $N$  is normal in  $G$  it is normal in any subgroup of  $G$ , so  $N$  is normal in  $KN$  and  $KN/N$  is a well-defined quotient group. Note further than any element of it is of the form  $(kn)N = k(nN) = kN$  (by associativity at the coset level and property 2 of the lemma on page 135). Hence  $\phi$  really has image in  $KN/N$ . It is a homomorphism as it is just the restriction to  $K$  of the natural homomorphism  $G \rightarrow G/N$ . It is onto as  $knN = kN$  has pre-image  $k \in K$ . The result follows from the First Isomorphism Theorem as  $\text{Ker}\phi = \{k \in K | kN = N\} = \{k \in K | k \in N\} = K \cap N$ .

- 38) Take  $\phi : G/N \rightarrow G/M$  given by  $gN \rightarrow gM$ . Now  $\phi$  is well-defined as if  $gN = hN$ , then  $gh^{-1} \in N$  so, as  $N$  is a subgroup of  $M$ , one sees that  $gh^{-1} \in M$  so  $gM = hM$ . Next  $\phi((gh)N) = (gh)M = gMhM = \phi(gN)\phi(hN)$ , so  $\phi$  is a homomorphism. It is clearly onto, as  $gN$  is the pre-image of  $gM$ . Finally,  $\text{Ker } \phi = \{gN | gM = M\} = \{gN | g \in M\} = M/N$ . Note again that  $M/N$  is a well-defined quotient group, as  $N$  being normal in the group  $G$  implies that it is normal in the subgroup  $M$  as well. The result now follows by applying the First isomorphism Theorem to  $\phi$ .
- 39) As in question 23, it is enough to know what  $\psi(1)$  is, where  $\psi$  is a given homomorphism. But  $\psi(1)$  generates a subgroup of  $\mathbb{Z}_k$ , so by Lagrange  $|\psi(1)|$  divides  $k$ , and it also divides  $|1|$  in  $\mathbb{Z}_n$ , i.e.  $n$ . So the number of homomorphisms is at most the number of elements in  $\mathbb{Z}_k$  with an order  $d$  which is a divisor of both  $n$  and  $k$ , i.e.  $\sum \phi(d)$ , where the sum is taken as stated in the question. Note finally that prescribing a homomorphism via  $\psi(1)$  is well-defined, i.e. single-valued, if  $\psi(1)$  divides both  $k$  and  $n$ , so the sum above is indeed the exact number of homomorphisms from  $\mathbb{Z}_n$  to  $\mathbb{Z}_k$ .
- 43)  $\phi : G \rightarrow \mathbb{Z}_{10}$  is onto, so by the First Isomorphism Theorem  $G/\text{Ker } \phi \cong \mathbb{Z}_{10}$ , so  $|G|/|\text{Ker}| = |\mathbb{Z}_{10}|$ , so 10 divides  $|G|$ .
- 49) Recall that an inner automorphism is of the form  $\phi_g(h) = ghg^{-1}$ . Define  $\Psi : G \rightarrow \text{Inn}(G)$  via  $g \rightarrow \phi_g$ . Then  $\Psi(gh) = \phi_{gh} = \phi_g\phi_h = \Psi(g)\Psi(h)$  (using the composition property of inner automorphisms). So  $\Psi$  is an homomorphism.  $\Psi$  is onto by its definition, and  $\text{Ker } \Psi = \{g \in G | \phi_g = \phi_e\} = \{g \in G | ghg^{-1} = h \text{ for any } h \in G\} = \{g \in G | gh = hg \text{ for any } h \in G\} = \mathcal{Z}(G)$ . Now the result follows by applying the First Isomorphism Theorem to  $\Psi$ .
- 55) Every non-identity element in such a group has one of the orders 5, 13, 65. Two subgroups  $H, K$  of prime order 13 have no nonidentity common elements, so  $|HK| = |H||K| = 13^2 > 65$ , hence there is at most one subgroup  $H$  of order 13, and so there must be elements of order 5. Now  $aHa^{-1}$  is a subgroup of the same order as  $H$ , so it must equal  $H$ , and this for any  $a \in G$ . Hence the normalizer  $N(H) = G$  (see Example 15 in the chapter). As  $H$  has prime order, it is cyclic, Abelian and so  $C(H)$  contains  $H$ . So 13 divides  $|C(H)|$  which divides 65. Hence either  $C(H) = G$  or  $C(H) = H$ . In the former case an element  $x = hf$ , with  $h$  a non-identity element in  $H$  and  $|f| = 5$  will have order 65. In the latter case,  $|N(H)/C(H)| = 65/13 = 5$ . But the  $N/C$  Theorem (Example 15) says that  $N(H)/C(H)$  is a subgroup of  $\text{Aut}(H)$ , but 5 does not divide  $|\text{Aut}(\mathbb{Z}_{13})| = |U(13)| = 12$ . The contradiction shows that the former possibility holds, and so  $G$  is cyclic.