Old midterm problems

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The following problems from old midterms are unfortunately mostly about duality, so they are *not* representive of most of our midterm material.

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1) Let V be the space of 2×2 matrices A satisfying $\mathrm{tr}(A) = 0$. Define elements $f_1, f_2, f_3 \in V^*$ by

$$f_1\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = a-2b+c, \quad f_2\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = a-b+c, \quad f_3\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = b-c,$$

for any $\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in V$.

a] Show that $\{f_1, f_2, f_3\}$ are linearly independent in V.

b] Find a dual basis to $\{f_1, f_2, f_3\}$. Express your answer as matrices, using the fact that $V^{**} \cong V$.

2) Let $V = \mathcal{P}_2(\mathbb{R})$. Define a map $T : V \to V^*$ by

$$(T(f))(p) = \int_0^1 f(t)p(t)dt.$$

a] Show that $T : V \to V^*$ is an isomorphism.

b] With V, T as in part (a), find T^t. That is, find an explicit formula for T^t using the isomorphism $V \cong V^*$.

3) Define elements $f_1, f_2, f_3 \in \mathcal{P}_2(\mathbb{R})^*$ by

$$f_1(p) = p(0), \quad f_2(p) = p(1), \quad f_3(p) = \int_0^1 p(x)dx, \quad \text{for any } p \in \mathcal{P}_2(\mathbb{R}).$$

a] Show that they are linearly independent in $\mathcal{P}_2(\mathbb{R})^*$.

b] Find a dual basis to $\{f_1, f_2, f_3\}$. Your answer can be expressed as polynomials, since $\mathcal{P}_2(\mathbb{R})^{**} \cong \mathcal{P}_2(\mathbb{R})$.

4) Let $V = M_{m\times n}(F)$. Define a map $T : M_{n\times m} \to V^*$ by $(T(B))(A) = \mathrm{trace}(BA)$.

a] Show that $T : M_{n\times m} \to V^*$ is an isomorphism.

b] Suppose $\{E_{ij}\}$ is the standrard basis of V. Let $\{E^*_{ij}\}$ denote the corresponding dual basis. Using part (a), express $\{E^*_{ij}\}$ as an $n \times m$ matrix, i.e. find the matrix C_{ij} so that $T(C_{ij}) = E^*_{ij}$. (Hint: write out the trace using the summation notation...).

5) V is a finite-dimensional vector space over a field F, and $B = \{\alpha_1, \dots \alpha_n\}$ is a basis of V.

a] Define what is meant by the dual space V^* of V, and the dual basis $B^* = \{\alpha_1^* \dots \alpha_n^*\}$ of V^*.

b] If $V = \mathbb{R}^2$ and $B = \{\alpha_1 = [1/2, 1/2]^t, \alpha_2 = [1/2, -1/2]^t\}$, describe $\alpha_i^*([x, y]^t)$, for $i = 1, 2$.

6) Perform the Gram-Schmidt process to obtain an orthonormal set (not just an orthogonal set) from each of the following sets of vectors.

a] In $\mathbb{R}^3$ with the usual dot product, vectors: $\left\{ \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \right\}$.

b] In $\mathcal{P}_2(\mathbb{C})$, with inner product $\langle f, g \rangle = f(-1)\overline{g(-1)} + f(0)\overline{g(0)} + f(1)\overline{g(1)}$, vectors: $\{i, x - i\}$.

7) In $\mathbb{R}^3$ with the standard inner product, find the projection of the vector $(2, -3, 1)$ onto the plane $2x + 2y - z = 0$.

8) If V is a finite dimenional inner product space, and $T \in \mathcal{L}(V)$, show that

$$N(T)^\perp = R(T^*).$$

9) For the following, if it is true, give a proof, if it is false, give counterexample.

- In a finite-dimensional inner product space V, suppose W is a subspace. Let P be the orthognal projection onto W. Then $P^2 = P$.