SOLUTIONS

ASSIGNMENT 1

(1) (a) False. A linear functional is a particular kind of linear transformation (a linear transformation from a vector space to its underlying field).

(b) True. Since a linear functional $f : F \to F$ is a linear transformation, it may be represented as a $(\dim F) \times (\dim F)$ matrix, and, trivially, $\dim F = 1$.

(c) True. The text proves that $\dim V = \dim V^*$ and a previous theorem says that two finite dimensional vector spaces (over the same field) are isomorphic iff they have the same dimension. Note that the truth of this statement is dependent on the assumption that all vector spaces are finite-dimensional.

(d) False. Dual vector spaces are vector spaces of functions (linear functionals), and so, in particular $\mathbb{R}^n$ is not the dual of some other vector space. Note that there is a difference between being isomorphic vector spaces (having "the same form") and being the *same* vector space. *N*ote the difference here from the text's answer.

(e) False. Consider the basis $\beta = \{2\}$ for $\mathbb{R}$, and let $T : \mathbb{R} \to \mathbb{R}^*$ be defined according to

$$T(a)(x) = ax.$$

A check determines that $T(\beta) = \{f\}$ where

$$f(x) = 2x.$$

However $\beta^* = \{g\}$ where

$$g(x) = g\left(\frac{x}{2} \cdot 2\right) = \frac{x}{2}.$$

(f) True. As $T : V \to W$, then $T^t : W^* \to V^*$, and so $\left(T^t\right)^t : V^{**} \to W^{**}$.

(g) True. Let $T : V \to W$ be an isomorphism. Using the results of EXERCISE 20 we see that $T^t : W^* \to V^*$ is both one-to-one and onto, and therefore T^t is an isomorphism. Or: one can simply compose some isomorphisms $f : V^* \to V$ and $g : W \to W^*$ (guaranteed to exist, see part (c)), with the give $h : V \to W$, so that $g \circ h \circ f$ provides an isomorphism.

(h) False. The derivative defines a *linear transformation*, but since the range of this transformation is a vector space (of functions), and not the underlying field, this transformation is not a linear functional.

(2) (a) This is a linear functional.

(b) This is not a linear functional, as the range is not the underlying field $\mathbb{R}$.

(c) This is a linear functional.

(d) This is not a linear functional, as it is not linear.

(e) This is a linear functional.

(f) This is a linear functional.

(3) (a) First note that each $(x, y, z) \in \mathbb{R}^3$, may be expressed as a linear combination of the β-basis elements as

$$(x, y, z) = \frac{2x - y}{2}(1, 0, 1) + \frac{y}{2}(1, 2, 1) + (z - x)(0, 0, 1).$$

Then $\beta^* = \{f_1, f_2, f_3\}$ where

$$\begin{aligned} g_1(x, y, z) &= \frac{2x - y}{2} \\ g_2(x, y, z) &= \frac{y}{2} \\ g_3(x, y, z) &= z - x. \end{aligned}$$

(b) It is easy to show that $\beta^* = \{g_1, g_2, g_3\}$ where

$$\begin{aligned} g_1(ax^2+bx+c) &= c \\ g_2(ax^2+bx+c) &= b \\ g_3(ax^2+bx+c) &= a. \end{aligned}$$

(4) We know that $\dim V^* = \dim V = 3$, and so if we can show that $\{f_1, f_2, f_3\}$ is linearly independent, we may conclude that it is a basis for V^*. So let $r, s, t \in \mathbb{R}$ be such that for any $(x, y, z) \in V$ we have

$$\begin{aligned} 0 &= (rf_1 + sf_2 + tf_3)(x, y, z) \\ &= rf_1(x,y,z) + sf_2(x,y,z) + tf_3(x,y,z) \\ &= r(x-2y) + s(x+y+z) + t(y-3z) \\ &= (r+s)x + (-2r+s+t)y + (s-3t)z. \end{aligned}$$

Picking $x = 1, y = z = 0$; and $x = z = 0, y = 1$; and $x = y = 0, z = 1$ we get the system of linear equations

$$\begin{array}{rcrcrcr} r & + & s & & & = & 0 \\ -2r & + & s & + & t & = & 0 \\ & & s & - & 3t & = & 0 \end{array}$$

which has only the trivial solution. Therefore the set is linearly independent.

If $\{(a,b,c), (r,s,t), (x,y,z)\}$ is the desired basis for V, then the following equalities must hold (by the properties of the dual basis):

$$\begin{array}{lll} f_1(a,b,c) = 1 & f_1(r,s,t) = 0 & f_1(x,y,z) = 0 \\ f_2(a,b,c) = 0 & f_2(r,s,t) = 1 & f_2(x,y,z) = 0 \\ f_3(a,b,c) = 0 & f_3(r,s,t) = 0 & f_3(x,y,z) = 1. \end{array}$$

Using the definitions of the functions f_1, f_2 and f_3 these equalities reduce to

$$\begin{array}{rrr} a - 2b = 1 & r - 2s = 0 & x - 2y = 0 \\ a + b + c = 0 & r + s + t = 1 & x + y + z = 0 \\ b - 2c = 0 & s - 3t = 0 & y - 3z = 1. \end{array}$$

Each of these columns is a system of linear equations, and solving gives the required basis:

$$\left\{ \left(\frac{2}{5}, \frac{-3}{10}, \frac{-1}{10}\right), \left(\frac{4}{5}, \frac{3}{10}, \frac{1}{10}\right), \left(\frac{1}{5}, \frac{1}{10}, \frac{-3}{10}\right) \right\}.$$

(5) It is easy to check that if $p(x) = ax + b$, then

$$\begin{aligned} f_1(p) &= \int_0^1 p(t)dt = \frac{a}{2} + b \\ f_2(p) &= \int_0^2 p(t)dt = 2a + 2b. \end{aligned}$$

Again, suppose that we have $r, s \in \mathbb{R}$ such that for any $p(x) = ax + b \in V$ we have

$$\begin{aligned} 0 &= (rf_1 + sf_2)(p) \\ &= rf_1(p) + sf_2(p) \\ &= r\left(\frac{a}{2} + b\right) + s(2a + 2b) \\ &= \left(\frac{r}{2} + 2s\right)a + (r + 2s)b. \end{aligned}$$

Picking the pairs $a = 2, b = 0$ and $a = 0, b = 1$ we get the system of linear equations

$$\begin{array}{ccccc} r & + & 4s & = & 0 \\ r & + & 2s & = & 0 \end{array}$$

and it is easy to check that this has only the trivial solution, and so $\{f_1, f_2\}$ is linearly independent, and by the same dimension argument, we may conclude that it is a basis for V^*. If $\{ax + b, rx + s\}$ is the desired basis for V, then the following equalities must hold (by the properties of the dual basis):

$$\begin{array}{ll} f_1(ax+b) = 1 & \qquad f_1(rx+s) = 0 \\ f_2(ax+b) = 0 & \qquad f_2(rx+s) = 1. \end{array}$$

Using the definitions of hte functions f_1, f_2 determined above, these equalities reduce to

$$\begin{array}{ll} \frac{a}{2} + b = 1 & \qquad \frac{r}{2} + s = 0 \\ 2a + 2b = 0 & \qquad 2r + 2s = 1. \end{array}$$

These two columns are systems of linear equations, and solving them gives the required basis

$$\left\{ -2x + 2, x - \frac{1}{2} \right\}.$$

(7) (a) For $p(x) = ax + b \in V$ we have

$$T^t(f)(p(x)) = f(T(ax+b)) = f(-2a - b, a + b) = (-2a - b) - 2(a + b) = -4a - 3b.$$

(b) Note that $\beta^* = \{f_1, f_2\}$ where $f_1(ax + b) = b$ and $f_2(ax + b) = a$, and that $\gamma^* = \{g_1, g_2\}$ where $g_1(a, b) = a$ and $g_2(a, b) = b$. Then

$$\begin{aligned} T^t(g_1)(ax+b) &= g_1 T(ax+b) = g_1(-2a - b, a + b) = -2a - b \\ T^t(g_2)(ax+b) &= g_2 T(ax+b) = g_2(-2a - b, a + b) = a + b. \end{aligned}$$

By inspection, we can see that

$$\begin{aligned} T^t(g_1) &= -f_1 - 2f_2 \\ T^t(g_2) &= f_1 + f_2 \end{aligned}$$

and therefore the matrix $[T^t]_{\gamma^*}^{\beta^*}$ is

$$[T^t]_{\gamma^*}^{\beta^*} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

(c) Simple calculations give

$$\begin{aligned} T(1) &= (1 - 2, 1 + 0) = (-1, 1) = -e_1 + e_2 \\ T(x) &= (0 - 2, 0 + 1) = (-2, 1) = -2e_1 + e_2 \end{aligned}$$

and therefore

$$[T]_\beta^\gamma = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

(13) (a) Trivially the zero function is in S^0. If $f, g \in S^0$ and $t \in F$, then for each $x \in S$ we have

$$(f + tg)(x) = f(x) + tg(x) = 0 + t \cdot 0 = 0$$

and thus $f + tg \in S^0$, and so S^0 is a subspace of V^*. $\square$

(b) Let $\{w_1, \ldots, w_n\}$ be a basis for W. Since $x \notin W$, then $\{w_1, \ldots, w_n, x\}$ is linearly independent, and therefore we may extend it to a basis

$$\beta = \{w_1, \ldots, w_n, x, v_1, \ldots, v_m\}$$

of V.
Define $f : V \to F$ by

$$f(a_1 w_1 + \ldots + a_n w_n + cx + b_1 v_1 + \ldots + b_m v_m) = c.$$

It is easy to show that $f \in W^0$ but $f(x) = 1 \neq 0$. ☐

(c) Recall that $\psi : V \to V^{**}$ is defined by $\psi(x)(f) = f(x)$. By THEOREM 2.26, since ψ is an isomorphism, for each $f \in V^{**}$ there is a (unique) $x \in V$ such that $f = \psi(x) = \hat{x}$.
Also, since ψ is a linear transformation, $\text{span}(\psi(S)) = \psi(\text{span}(S))$. Therefore, it suffices to show that for all $x \in V$

$$\hat{x} \in \left(S^0\right)^0 \Leftrightarrow x \in \text{span}(S).$$

If $x \in \text{span}(S)$, then there are $x_1, \ldots, x_n \in S$ and $a_1, \ldots, a_n \in F$ such that $x = a_1 x_1 + \ldots + a_n x_n$. Then, for each $f \in S^0$ we have

$$\hat{x}(f) = f(x) = f(a_1 x_1 + \ldots + a_n x_n) = a_1 f(x_1) + \ldots + a_n f(x_n) = 0$$

by definition of S^0. Therefore, $\hat{x} \in \left(S^0\right)^0$.
If $x \notin \text{span}(S)$, then by (b) there is an $f \in (\text{span}(S))^0$ such that $f(x) \neq 0$. Therefore $\hat{x}(f) = f(x) \neq 0$, and thus it cannot be that $\hat{x} \in \left(S^0\right)^0$. ☐

(14) Working according to the hint, let $\{x_1, \ldots, x_k\}$ be a basis for W, and extend it to a basis $\beta = \{x_1, \ldots, x_n\}$ for V. Consider the dual basis $\beta^* = \{f_1, \ldots, f_n\}$. We now wish to show that $\{f_{k+1}, \ldots, f_n\}$ is a basis for W^0.

Trivially, it is linearly independent, so we need only check that it spans. It is also clear that each of $f_{k+1}, \ldots, f_n$ is in W^0. Let $f \in W^0$, and express it as
$f = a_1 f_1 + \ldots + a_k f_k + a_{k+1} f_{k+1} + \ldots + a_n f_n$ for some scalars $a_1, \ldots, a_n$. By (a), as W^0 is a subspace, it follows that

$$a_1 f_1 + \ldots + a_k f_k = f - (a_{k+1} f_{k+1} + \ldots + a_n f_n) \in W^0.$$

However, for each $i = 1, \ldots k$ we have that

$$0 = (a_1 f_1 + \ldots + a_k f_k)(x_i) = a_i f_i(x_i) = a_i$$

and thus $f \in \text{span}\{f_{k+1}, \ldots, f_n\}$, as required.

(20) (a) As we will be using the result from EXERCISE 19 for this, we will first answer that question.

Lemma 0.1 (Exercise 19). *Let V be a nonzero vector space, and let W be a proper subspace of V. Prove that there exists a nonzero linear functional $f \in V^*$ such that $f(x) = 0$ for all $x \in W$.*

Proof. Using the results of SECTION 1.7, we may begin with a basis β for W, and then extend it to a basis γ of V (as β is a linearly independent subset of V, we may use the REPLACEMENT THEOREM). As W is a proper subspace, then $\beta \subsetneqq \gamma$.
We then define a function $g : \gamma \to F$ as follows:

$$g(x) = \begin{cases} 1, & \text{if } x \notin \beta \\ 0, & \text{if } x \in \beta. \end{cases}$$

Using EXERCISE 34 in SECTION 2.1 (31 for third edition), there is a unique linear transformation $f : V \to F$ (a linear functional) such that $f(x) = g(x)$ for all $x \in \gamma$.

For any $x \in \gamma$ which is not in β, we have $f(x) = g(x) = 1$, and so f is a nonzero linear functional.
For any $y \in W$, there are $y_1, \ldots, y_n \in \beta$ and scalars $a_1, \ldots, a_n$ such that $y = a_1 y_1 + \ldots + a_n y_n$. Then we have

$$f(y) = f(a_1 y_1 + \ldots + a_n y_n) = a_1 f(y_1) + \ldots + a_n f(y_n) = 0.$$

Therefore f is as required. □

If T is onto, suppose that $f \in W^*$ is such that $T^t(f) = fT = 0$. For any $w \in W$ there is a $v \in V$ such that $T(v) = w$, since T is onto, and thus $f(w) = f(T(v)) = 0$. It then follows that $f = 0$, and so T^t is one-to-one.
If T is not onto, then $R(T)$ is a proper subspace of W, and using EXERCISE 19 there is a nonzero linear functional $f \in W^*$ such that $f(y) = 0$ for all $y \in R(T)$. I claim that $f \in N(T^t)$, that is for all $x \in V$ we have $(T^t(f))(x) = 0$. Letting $x \in V$, we have

$$\begin{aligned}\left(T^t(f)\right)(x) &= fT(x) && \text{(by definition of } T^t\text{)}\\ &= f(T(x)) && \\ &= 0 && \text{(as } T(x) \in R(T)\text{)}\end{aligned}$$

As $N(T^t) \neq 0$, then T^t is not one-to-one. □

(b) To prove this, will will instead show the following: If $\psi : V \to V^{**}$ and $\varphi : W \to W^{**}$ are as in THEOREM 2.26, then

$$\varphi T \psi^{-1} = \left(T^t\right)^t.$$

If we can accomplish this, then using (a) and the fact that ψ and φ are isomorphisms, we get that T is one-to-one iff $\varphi T \psi^{-1} = (T^t)^t$ is one-to-one iff T^t is onto.
Let $f \in V^{**}$. We must then show that $\left(\varphi T \psi^{-1}\right)(f) = (T^t)^t(f)$. First, let $\psi^{-1}(f) = x$, and take $g \in W^*$.
Then the left-hand-side equals

$$\begin{aligned}\left(\left(\varphi T \psi^{-1}\right)(f)\right)(g) &= \left((\varphi T)\left(\psi^{-1}(f)\right)\right)(g) = ((\varphi T)(x))(g) = (\varphi(T(x)))(g)\\ &= \widehat{T(x)}(g) = g\left(T(x)\right) = gT(x)\end{aligned}$$

while the right-hand-side equals

$$\left(\left(T^t\right)^t(f)\right)(g) = \left(fT^t\right)(g) = f\left(T^t(g)\right) = \hat{x}\left(gT\right) = \hat{x}\left(gT\right) = gT(x).$$

As the two expressions are equal, and we are done. □

ASSIGNMENT 2

Section 6.2

15)a) Using Theorem 6.5

$$\begin{aligned}\langle x, y\rangle &= \langle \sum_{i=1}^{n} \langle x, v_i\rangle v_i, \sum_{j=1}^{n} \langle y, v_j\rangle v_j \rangle\\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_j\rangle} \langle v_i, v_j\rangle\\ &= \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_j\rangle} \delta_{ij} = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.\end{aligned}$$

16)a) The set $S \cup \{x\}$ spans a subspace of dimension $m = n$ or $m = n + 1$. In any case, using Paresval's identity for this subspace with an orthonormal basis of $\{v_1 \dots v_n \dots v_m\}$ vectors (which extends S):

$$||x||^2 = \langle x, x\rangle = \sum_{i=1}^{m} \langle x, v_i\rangle \overline{\langle x, v_i\rangle} = \sum_{i=1}^{m} |x, v_i|^2 \geq \sum_{i=1}^{n} |x, v_i|^2.$$

ASSIGNMENT 3

Section 6.3

7) Take L_A to be multiplication by the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. in $\mathbb{R}^2$ Its null space is $\mathrm{Span}((1,0)^t)$, while L_A^* has a matrix with respect to the standard basis given by the transpose $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and its null-space is $\mathrm{Span}((0,1)^t)$.

Section 6.4

10) The left hand side is equal to

$$\begin{aligned}
\langle T(x) \pm ix, T(x) \pm ix\rangle &= ||T(x)||^2 + ||X||^2 \pm \langle T(x), ix\rangle \pm \langle ix, T(x)\rangle \\
&= ||T(x)||^2 + ||X||^2 \mp i\langle T(x), x\rangle \pm i\langle x, T(x)\rangle \\
&= ||T(x)||^2 + ||X||^2 \mp i\left(\langle T(x), x\rangle - \langle x, T(x)\rangle\right) \\
&= ||T(x)||^2 + ||X||^2, \quad \text{since } T \text{ is self-adjoint.}
\end{aligned}$$

The equality implies that if $(T - iI)(x)$ is zero, its norm is zero, and so, as a sum of squares, both the norms of $T(x)$ and of x are zero, but this means that $x = 0$. Hence $N(T - iI) = \{0\}$, so $T - iI$ is one-to-one, and so, on a finite dimensional vector space, it is onto (as the dimension of the range is the dimension of V). So it is an invertible operator. Now if L is an invertible operator, the equation $LL^{-1} = I$ implies, after taking the adjoint on both sides, that $(L^{-1})^* = (L^*)^{-1}$. Therefore, in this case the adjoint of $(T - iI)^{-1}$ is the inverse of $(T - iI)^* = T^* + iI = T + iI$.