

Deriving Einstein's Field Equation From a Least Action Principle

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1 Introduction

The expression of physical laws by employing a variational principle is used extensively in research both for its simplicity and elegance as well as for its usefulness in gaining further insights. In particular, the principle of least action in classical mechanics allows one to conveniently obtain the equations of motion for even the most complicated of systems. Associated to each system of particles is a real-valued function, L , of the trajectories $q_i(t)$ of the particles, the velocities $\dot{q}_i(t)$, and time, t . Here $q_i(t) = x_i^\mu(t)$, $\mu = 1, 2, 3$. L is called the Lagrangian and is typically equal to $T - V$, where T and V are the total kinetic and potential energies of the system, respectively. In the time interval $[t_1, t_2]$, the action, S , is defined as

$$S = \int_{t_1}^{t_2} L(q_1(t), \dots, q_N(t), \dot{q}_1(t), \dots, \dot{q}_N(t), t) dt \quad (1)$$

According to the principle of least action, the actual trajectory traversed by the system of particles in $[t_1, t_2]$ is the one for which S is a minimum. This trajectory is obtained by evaluating the first order change, δS , of the action as a result of varying the trajectories, $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$ subject to the condition that the varied paths all meet at t_1 and t_2 so that $\delta q_i(t) \equiv 0$ at t_1 and t_2 . Setting $\delta S \equiv 0$ (S is extremal) for all $\delta q_i(t)$, one obtains a set of N differential equations for $q_i(t)$, the actual trajectories of the N particles. These are the Euler-Lagrange equations of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}, \quad i = 1, 2, \dots, N \quad (2)$$

These equations are equivalent to the Newtonian equations of motion, which are based on the Hamiltonian formulation of classical mechanics.

The above variational approach not only applies to trajectories of individual particles but also to continuous spacetime fields, such as the metric in general relativity.

The Lagrangian formulation of fundamental physics is also important for developing quantum field versions of classical field theories. These classical

field theories are often expressed in terms of a field equation involving a field of force. One must do away with a force field in developing a quantized version of the physics and the Lagrangian formulation of quantum mechanics is a necessary tool since no fields are involved. For example, Richard Feynman developed a Lagrangian formulation of quantum mechanics (the path integral formulation) and used it in developing quantum electrodynamics: the quantum field theory of the classical electromagnetism of Maxwell. Likewise, a Lagrangian formulation of general relativity is important for formulating a quantum theory of gravity.

We now generalize the least action principle to a Lagrangian formulation of a field theory and apply it to the case of obtaining Einstein's field equation of general relativity.

2 Lagrangian Formulation of a Field Theory

In the case of a field theory in flat Minkowski spacetime, the trajectory (for a single particle) $q(t)$ is replaced by a set of spacetime-dependent fields, $\Phi^i(x^\mu)$, $\mu = 0, 1, 2, 3$, over a spacetime manifold, M . Each Φ^i is a function on spacetime, in some coordinate system, and the index i labels the individual fields.

Here, we may express the Lagrangian as an integral of the Lagrangian density $\ell(\Phi^i, \partial_\mu \Phi^i)$, a scalar:

$$L = \int \ell d^3x. \quad (3)$$

Thus, the action is

$$S = \int \ell dt = \int_U \ell d^4x, \quad (4)$$

where U is a compact subspace of M . We now perturb Φ^i and $\partial_\mu \Phi^i$ by a small variation:

$$\Phi^i \rightarrow \Phi^i + \delta\Phi^i, \quad (5)$$

$$\partial_\mu \Phi^i \rightarrow \partial_\mu \Phi^i + \delta(\partial_\mu \Phi^i) = \partial_\mu \Phi^i + \partial_\mu(\delta\Phi^i). \quad (6)$$

Hence, for the Lagrangian density, we have:

$$\ell(\Phi^i, \partial_\mu \Phi^i) \rightarrow \ell(\Phi^i + \delta\Phi^i, \partial_\mu \Phi^i + \partial_\mu \delta\Phi^i) \quad (7)$$

$$\approx \ell(\Phi^i, \partial_\mu \Phi^i) + \frac{\partial \ell}{\partial \Phi^i} \delta\Phi^i + \frac{\partial \ell}{\partial (\partial_\mu \Phi^i)} \partial_\mu (\delta\Phi^i). \quad (8)$$

Substituting this for the action, we have:

$$S \rightarrow S + \delta S = S + \int_U d^4x \left[\frac{\partial \ell}{\partial \Phi^i} \delta \Phi^i + \frac{\partial \ell}{\partial (\partial_\mu \Phi^i)} \partial_\mu (\delta \Phi^i) \right]. \quad (9)$$

Integrating the last term by parts, we get:

$$\int_U d^4x \frac{\partial \ell}{\partial (\partial_\mu \Phi^i)} \partial_\mu (\delta \Phi^i) = \int_U d^4x \partial_\mu \left(\frac{\partial \ell}{\partial (\partial_\mu \Phi^i)} \delta \Phi^i \right) - \int_U d^4x \partial_\mu \left(\frac{\partial \ell}{\partial (\partial_\mu \Phi^i)} \right) \delta \Phi^i. \quad (10)$$

Now, by Stokes' theorem, the first term on the right hand side of (10) vanishes because, as in the case of a single particle trajectory, the variation $\delta \Phi^i$ vanishes on the boundary of U. Thus, plugging (10) into (9), we have:

$$\delta S = \int_U d^4x \left[\frac{\partial \ell}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \ell}{\partial (\partial_\mu \Phi^i)} \right) \right] \delta \Phi^i. \quad (11)$$

The functional derivative $\frac{\delta S}{\delta \Phi^i}$ of a functional S with respect to a function Φ^i is defined by $\delta S = \int d^4x \frac{\delta S}{\delta \Phi^i} \delta \Phi^i$. Thus, S is extremal when the functional derivative vanishes.

If Φ^i are the actual fields, then $\delta S \equiv 0$ for all $\delta \Phi^i$ small. Thus, the above integrand must vanish. Therefore, the equations governing our field theory are:

$$\partial_\mu \left(\frac{\partial \ell}{\partial (\partial_\mu \Phi^i)} \right) = \frac{\partial \ell}{\partial \Phi^i}. \quad (12)$$

3 Deriving the Einstein Field Equation

In general relativity, the field equation is the Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (13)$$

where $R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda$ is the Ricci tensor, $g_{\mu\nu}$ is the metric, $T_{\mu\nu}$ is the energy-momentum tensor, and $R = g^{\mu\nu} R_{\mu\nu}$ is the trace of the Ricci tensor, the Ricci scalar.

We shall now show that (13) may be derived from a least action principle analogous to the case presented in section 2.

In curved spacetime, the Lagrangian is a function of the fields and their covariant derivatives. Thus, in local coordinates,

$$S = \int \ell(\Phi^i, \nabla_\mu \Phi^i) d^4x. \quad (14)$$

In general relativity the metric, $g_{\mu\nu}$, is the dynamical variable, defined on a four dimensional manifold, M . Therefore, the action must be a functional of it. We suppose that the variation of the metric vanishes outside of U and on its boundary.

Due to invariance of S under general coordinate transformations, the Lagrangian density must be a scalar density. Since $\nabla_\alpha g_{\mu\nu} = 0$ and the first derivatives can all be transformed to zero at any one point, the Lagrangian density must involve at least second derivatives of the metric.

Now, the Riemann tensor,

$$R_{\beta\gamma\eta}^\alpha = \partial_\beta \Gamma_{\gamma\eta}^\alpha - \partial_\gamma \Gamma_{\beta\eta}^\alpha + \Gamma_{\gamma\eta}^\sigma \Gamma_{\beta\sigma}^\alpha - \Gamma_{\beta\eta}^\sigma \Gamma_{\gamma\sigma}^\alpha \quad (15)$$

is made from second derivatives of the metric and any tensor made from product of the metric and its first and second derivatives can be expressed in terms of the metric and the Riemann tensor. Since the only independent scalar constructed from the Riemann tensor is the Ricci scalar, R , the only independent scalar constructed from the metric with derivatives no higher than second order, is R .

Thus we propose, as Hilbert did, the Lagrangian $R \text{ vol}$, where vol is the volume form associated to g . In local coordinates, $\text{vol} = \sqrt{|\det(g_{\mu\nu})|} d^4x$. We set $\det(g_{\mu\nu}) = g < 0$. Thus $\text{vol} = \sqrt{-g} d^4x$. We thus form the Einstein-Hilbert action (although it was discovered by Hilbert and he used it to derive Einstein's field equation five days before Einstein published the field equation)

$$S_H = \int_U R \sqrt{-g} d^4x. \quad (16)$$

Therefore,

$$\delta S_H = \int_U (\delta R) \sqrt{-g} d^4x + \int_U R (\delta \sqrt{-g}) d^4x. \quad (17)$$

To calculate the second integral, we use the following fact for any square matrix M with nonzero determinant: $\ln(\det(M)) = \text{tr}(\ln(M))$, where $\ln(M)$ is defined by $e^{\ln(M)} = M$.

The variation of both sides of this yields $\frac{1}{\det(M)} \delta(\det(M)) = \text{tr}(M^{-1} \delta M)$. Taking $M = g_{\mu\nu}$, we get: $\frac{1}{g} \delta g = g^{\mu\nu} \delta g_{\mu\nu} = -g_{\mu\nu} \delta g^{\mu\nu}$, using $g^{\alpha\beta} g_{\alpha\beta} = 4$, for which the variation (using the product rule) is zero. Thus, $\delta g = -g(g_{\mu\nu} \delta g^{\mu\nu})$.

Therefore,

$$\delta \sqrt{-g} = \frac{-1}{2\sqrt{-g}} \delta g = \frac{1}{2} \frac{g}{\sqrt{-g}} g_{\mu\nu} \delta g^{\mu\nu} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \quad (18)$$

Now,

$$\delta R = (\delta g^{\mu\nu}) R_{\mu\nu} + g^{\mu\nu} (\delta R_{\mu\nu}) \quad (19)$$

It is convenient to carry out the variation of the action with respect to the inverse metric, which we can do since stationary points of the action with respect to variations in the metric are equivalent to those with respect to the inverse metric: $g^{\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{\alpha\beta}$. Thus, we need only calculate the second term in (19).

Recall the definition of the Christoffel symbols:

$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\eta}(\partial_\beta g_{\gamma\eta} + \partial_\gamma g_{\beta\eta} - \partial_\eta g_{\beta\gamma})$. From this, we compute the variation:

$$\delta\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\eta}(\nabla_\beta\delta g_{\gamma\eta} + \nabla_\gamma\delta g_{\beta\eta} - \nabla_\eta\delta g_{\beta\gamma}). \quad (20)$$

Now, from the definition, (15), of the Riemann tensor, the variation is: $\delta R_{\beta\gamma\eta}^\alpha = \nabla_\beta\delta\Gamma_{\gamma\eta}^\alpha - \nabla_\gamma\delta\Gamma_{\beta\eta}^\alpha$. Therefore, the variation of the Ricci tensor is: $\delta R_{\alpha\beta} = \delta R_{\alpha\gamma\beta}^\gamma = \nabla_\alpha\delta\Gamma_{\gamma\beta}^\gamma - \nabla_\gamma\delta\Gamma_{\alpha\beta}^\gamma$. Upon substituting (20), we get: $\delta R_{\alpha\beta} = \frac{1}{2}(g^{\gamma\eta}\nabla_\alpha\nabla_\beta\delta g_{\gamma\eta} + g^{\gamma\eta}\nabla_\gamma\nabla_\eta\delta g_{\alpha\beta} - g^{\gamma\eta}\nabla_\gamma(\nabla_\beta\delta g_{\alpha\eta} + \nabla_\alpha\delta g_{\beta\eta}))$. Upon substituting this into (19), we get:

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla^\lambda\nabla_\lambda(g^{\mu\nu}\delta g_{\mu\nu}) - \nabla^\mu\nabla^\nu\delta g_{\mu\nu} \quad (21)$$

$$= R_{\mu\nu}\delta g^{\mu\nu} + \nabla^\mu(g^{\gamma\eta}\nabla_\mu\delta g_{\gamma\eta} - \nabla^\nu\delta g_{\mu\nu}) \quad (22)$$

$$= R_{\mu\nu}\delta g^{\mu\nu} + \nabla^\mu\omega_\mu. \quad (23)$$

We now substitute (18) and (23) into (17), thus giving

$$\delta S_H = \int_U [R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}]\delta g^{\mu\nu} vol + \int_U \nabla^\mu\omega_\mu vol. \quad (24)$$

The second term in (24) can be ignored. It is generally argued that this term is zero because, using Stokes' theorem, ω_μ vanishes on the boundary of U. Although, by assumption, the variations of the inverse metric vanish on the boundary of U, its covariant derivatives may not vanish. In more rigorous derivations, one may either use a variational principle so that both the metric and the Christoffel symbols (i.e. the connection) are independently varied (as in the Palatini action), or one may include a boundary term in the Einstein-Hilbert action to cancel the $\nabla^\mu\omega_\mu$ term in (24).

Thus, S_H is extremal if $\delta S_H \equiv 0$ for all $\delta g^{\mu\nu}$ which is true only if $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$, which is Einstein's field equation in vacuum (zero energy-momentum tensor).

Since we dealt only with the gravitational part of the action, S_H , neglecting matter fields, we arrived at Einstein's equation in vacuum. For the complete field equation, (13), we consider the following action:

$$S = \frac{1}{16\pi G}S_H + S_M, \quad (25)$$

where S_M is the action for matter.

Recall that the functional derivative of the action satisfies:

$$\delta S = \int_U d^4x \frac{\delta S}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = \int_U d^4x \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \delta g^{\mu\nu} \sqrt{-g}. \quad (26)$$

Thus, in our case:

$$\delta S = \frac{1}{16\pi G} \delta S_H + \delta S_M \quad (27)$$

$$= \int_U \frac{1}{16\pi G} [R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}] \delta g^{\mu\nu} vol + \int_U \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \delta g^{\mu\nu} vol. \quad (28)$$

Therefore, for S to be extremal, we must have:

$$\frac{1}{16\pi G} [R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}] = - \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \quad (29)$$

Defining

$$T_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \quad (30)$$

we arrive at Einstein's field equation, (13): $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$.

In some sense, (30) is in fact the energy-momentum tensor only because it is a symmetric, conserved, (0,2) tensor with dimensions of energy density.

4 Conclusions

Fundamental physical laws of nature may be derived from a least action principle using an appropriate Lagrangian. General relativity is no different, as we have seen. One may also derive the field equation via a variational principle whereby both the metric and the connection are varied. The resulting action is called the Palatini action. Among the advantages of using a Lagrangian formulation for any theory are that conserved quantities can easily be obtained by looking for symmetries of the Lagrangian (according to Noether's theorem, every symmetry of a Lagrangian implies the existence of a conservation law) and a Lagrangian formulation of a classical field is a necessary first step when constructing a quantized version of the field. In the case of general relativity, a Lagrangian formulation is very important in the quantization of gravity, which eludes physicists to this day.

5 Bibliography

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