

Existence of Lorentzian Metrics on Compact  
Manifolds and the Euler Characteristic

Tim Tzaneteas

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# 1 Introduction

The theory of General Relativity places restrictions on which manifolds can be models for spacetime, and one of these restrictions is that the manifold be equipped with a smooth Lorentzian metric. This leads naturally to the question of which manifolds admit such a metric, since any that do not cannot be models for spacetime. The question is non-trivial since it is known that there are manifold, such as  $S^4$  which do not admit a Lorentzian metric.

The best possible answer to a question like this would be one that equates the existence of a Lorentzian metric with some easily calculable topological property of the manifold. This is what I do in this paper, where I show that for compact manifolds, a smooth Lorentzian metric exists if and only if the Euler characteristic of the manifold is zero, or more formally:

**Theorem 1.1.** *Let  $M$  be a compact smooth manifold. Then  $M$  admits a Lorentzian metric if and only if the Euler characteristic of  $M$ ,  $\chi(M)$ , is identically zero.*

This equivalence will not be proven directly, but indirectly through the concept of a field of line elements, which is defined as follows:

**Definition 1.2.** Let  $M$  be a smooth manifold. A field of line elements  $X$  is a choice at each point  $p \in M$  of an equivalence class of  $T_p M$ , where the equivalence relation is  $v \sim \lambda v$ ,  $\lambda \neq 0$ .

Clearly, a field of line elements determines a vector field (simply choose an element of each equivalence class) and vice versa, so in what follows, the two concepts are used almost interchangeably.

The basic idea of the paper is to show that both the existence of a Lorentzian metric and the Euler characteristic's vanishing are equivalent to the existence of a nowhere zero field of line elements, and thus equivalent to each other.

## 2 Lorentzian metrics

In this section I will prove the equivalence of the existence of a Lorentzian metric to the existence of a non-vanishing field of line elements. First a lemma.

**Lemma 2.1.** *Every manifold  $M$  admits a Riemannian metric.*

*Proof.* Let  $\{(U_\alpha, \phi_\alpha)\}$  be a covering of  $M$  by coordinate charts and let  $\{f_\alpha\}$  be a partition of unity subordinate to  $\{U_\alpha\}$ . Then let  $\langle, \rangle$  be the standard inner product on  $\mathbb{R}^n$ , where  $n$  is the dimension of  $M$ . Fixing  $p \in M$ , we can then, for  $v, w \in T_p M$ , define  $g$  on  $M$  by

$$g(v, w) = \sum_{\alpha} f_{\alpha}(p) \langle (d\phi_{\alpha})_p(v), (d\phi_{\alpha})_p(w) \rangle$$

where  $g$  is well-defined because at  $p$ , only finitely many of the  $f_{\alpha}$ 's are non-zero.

From the fact that  $\langle, \rangle$  is an inner product, it follows that  $g$  is symmetric and bilinear and, since each  $f_\alpha$  is positive, we also have that  $g(v, v) \geq 0$  for all  $v$ . So, to show  $g$  is a Riemannian metric, all that is left is to show that if  $g(v, v) = 0$ , then  $v = 0$ . So assume that that is the case. Since  $\sum_\alpha f_\alpha(p) = 1$ , there is some  $f_\alpha$  such that  $f_\alpha(p) > 0$ . It then follows that  $\langle (d\phi_\alpha)_p(v), (d\phi_\alpha)_p(v) \rangle = 0$ , so  $(d\phi_\alpha)_p(v) = 0$ . But  $(d\phi_\alpha)_p$  is an isomorphism, so  $v = 0$ .  $\square$

Now the main result of this section.

**Theorem 2.2.** *A manifold  $M$  admits a Lorentzian metric if and only if  $M$  admits a non-vanishing field of line elements.*

*Proof.* First, suppose  $M$  has a non-vanishing field of line elements. Let  $g$  be a Riemannian metric on  $M$ , whose existence is guaranteed by the preceding lemma. Fix  $p \in M$ , and let  $u$  be a unit vector contained in the line element at  $p$ . Then, for  $v, w \in T_pM$ , define  $\hat{g}$  by

$$\hat{g}(v, w) = g(v, w) - 2g(u, v)g(u, w)$$

It follows at once that  $\hat{g}$  is bilinear and symmetric. Now suppose  $\hat{g}(v, w) = 0$  for all  $w \in T_pM$ . Then taking  $w = u$ , we get  $g(v, u) = 2g(u, v)g(u, u) = 2g(u, v)$ , since  $g(u, u) = 1$ , and thus  $g(u, v) = 0$ . If we then take  $w = v$ , we have  $g(v, v) = 2(g(u, v))^2 = 0$ , so  $v = 0$  because  $g$  is a Riemannian metric. This means  $\hat{g}$  is non-degenerate and thus a metric.

To see that  $\hat{g}$  is in fact Lorentzian, write, in local coordinates,  $g(v, w) = g_{ab}v^a w^b$  so

$$(g(u, v))^2 = (g_{ab}u^a v^b)(g_{cd}u^c v^d) = u_b u_d v^b v^d$$

and thus

$$\hat{g}(v, v) = g(u, u) - 2(g(u, v))^2 = (g_{ab} - u_a u_b)v^a v^b$$

Now let  $(e_i)_{i=1}^n$  be an orthonormal basis for  $g$  at  $p$  such that  $e_1 = u$ . Then  $u^1 = u_1 = 1$  and  $u^a = u_a = 0$  for  $a = 2, 3, \dots, n$ . Also  $g_{ab} = \delta_{ab}$ , so that

$$\begin{aligned} \hat{g}(v, v) &= \sum_{a=1}^n \sum_{b=1}^n (2u_a u_b - g_{ab})v^a v^b \\ &= \sum_{a=1}^n (2u_a v^a v^1 - \sum_{b=1}^n g_{ab}v^a v^b) \\ &= 2v^1 v^1 - \sum_{a=1}^n \sum_{b=1}^n g_{ab}v^a v^b \\ &= (v^1)^2 - (v^2)^2 - \dots - (v^n)^2 \end{aligned}$$

This shows that  $\hat{g}$  is in fact a Lorentzian metric.

Conversely, suppose  $\hat{g}$  is a Lorentz metric on  $M$ , and again let  $g$  be a Riemannian metric on  $M$ . Consider the equations

$$(\hat{g}_{ab} - \lambda g_{ab})v^b = 0$$

The solutions are eigenvectors corresponding to the eigenvalues determined by  $\det[\hat{g}_{ab} - \lambda g_{ab}] = 0$ . By choosing an orthonormal basis for  $g$ , we can find a matrix  $T$  such that  $T^{-1}[g_{ab}]T = I$ , the identity. Since  $[g_{ab}]$  is unitary it follows that the the transpose of  $T$ ,  $T^t$ , is equal to  $T^{-1}$ . So if we set  $A = T^{-1}[\hat{g}_{ab}]T$ , then  $A^t = T^t[\hat{g}_{ab}]^t(T^{-1})^t = T^{-1}[\hat{g}_{ab}]T = A$ , i.e.  $A$  is symmetric. Since the eigenvalues we are looking for are also the eigenvalues of  $\det[A - \lambda I] = 0$ , it follows from the fact that  $A$  is symmetric, that the eigenvalues are real.

We can then find a system of coordinates in which  $\hat{g}(v, v) = \lambda_1(v^1)^2 + \dots + \lambda_n(v^n)^2$ . Because  $\hat{g}$  is Lorentzian, one, and only one, of the eigenvalues will always be strictly positive. By selecting the line element determined by the vector corresponding to that positive eigenvalue, we get a smooth nowhere-zero field of line elements on  $M$ .  $\square$

### 3 Euler Characteristic

In this section I will show that if a manifold's Euler characteristic is zero, then the manifold admits a non-zero field of line elements.

First, I will introduce the concept of index.

**Definition 3.1.** (a) Let  $U \subset \mathbb{R}^n$  be open and let  $V$  be a smooth vector field on  $U$  with an isolated zero at the point  $z \in U$ . Consider the function

$$f(x) = \frac{V_x}{\|V_x\|}.$$

The *index*  $\iota$  of  $V$  at the zero  $z$  is defined to be the degree of the map  $f$ .

(b) Let  $M$  be an  $n$ -dimensional manifold and  $V$  be a smooth vector field on  $M$  with an isolated zero at  $p \in M$ . The *index*  $\iota$  of  $V$  at the zero  $p$  is defined to be the index of  $(d\phi) \circ V \circ \phi^{-1}$  at  $\phi(p)$ , where  $\phi : U \rightarrow \mathbb{R}^n$  is any coordinate chart with  $p \in U$ .

(c) For the situation in (b), the *index* of  $V$  is defined as

$$\iota(V) = \sum \iota$$

where the sum is over all zeroes of  $V$ .

A useful property of the index that we will require later is the following.

**Theorem 3.2.** *Let  $V$  be a smooth vector field on some open subset  $U \subset \mathbb{R}^n$ , and let  $V$  have a single zero at  $z$  with index 0. Then there is a smooth vector field  $\tilde{V}$  on  $U$  that has no zeroes and agrees with  $V$  outside of some compact subset containing  $z$ .*

*Proof.* We can assume without loss of generality that  $z = 0$ . Consider the map  $f : S^{n-1} \rightarrow S^{n-1}$  defined by

$$f(x) = \frac{\epsilon V_x}{\|\epsilon V_x\|}$$

where  $\epsilon > 0$  is chosen so that  $\epsilon S^{n-1} \subset U$ .

Since the index at  $z$  is zero, the degree of  $f$  is zero. Since  $f$  is a map from the  $(n-1)$ -sphere to itself, this means that  $f$  is not surjective. So there is  $y_0 \in S^{n-1}$  such that  $f^{-1}\{y_0\} = \emptyset$ . Taking this back to  $V$ , this means  $V$  is never directed in the direction of  $y_0$ .

Now choose a smooth function  $g : U \rightarrow \mathbb{R}$  such that

$$g(x) = \begin{cases} 1 & \text{if } \|x\| \leq \frac{\epsilon}{2} \\ 0 & \text{if } \|x\| \geq \epsilon \end{cases}$$

Then define  $\tilde{V}$  by

$$\tilde{V}_x = V_x - g(x)y_0$$

Then  $\tilde{V}$  agrees with  $V$  outside the closed ball of radius  $\epsilon$  and inside has no zeros since  $V_x$  never points in the direction of  $y_0$ , and this means  $\tilde{V}$  is what we want.  $\square$

A main result in this area, which I will use without proof, is the Poincaré-Hopf Theorem which equates the index of a vector field with the Euler characteristic of the manifold.

**Theorem 3.3 (Poincaré-Hopf Theorem).** *Let  $M$  be a compact manifold. Then for any smooth vector field  $V$  on  $M$ , we have*

$$\iota(V) = \chi(M).$$

A consequence of this theorem is that all vector fields on a manifold have the same index and this in turn lets us define the index of a field of line elements.

Now we turn to constructing a nowhere zero field of line elements on a manifold whose Euler characteristic is zero. For this we need the following definitions.

**Definition 3.4.** (a) The *standard  $p$ -simplex* is the set

$$\{(x_1, \dots, x_p) \in \mathbb{R}^n \mid x_i \geq 0 \text{ and } x_1 + \dots + x_p = 1\}.$$

Its  $(p-1)$ -faces are

$$\{(x_1, \dots, x_p) \in \mathbb{R}^n \mid x_j = 0, x_i \geq 0 \text{ and } x_1 + \dots + x_p = 1\}, j = 1, \dots, p.$$

- (b) A  *$p$ -simplex* on a manifold  $M$  is the image of the standard  $p$ -simplex under a smooth map from an open subset of  $\mathbb{R}^n$  to  $M$ . Its  $(p-1)$ -faces are the images of the standard  $p$ -simplex's  $(p-1)$ -faces under the same smooth map.
- (c) A *complex* on  $M$  is a finite formal sum of simplices.
- (d) Let  $M$  be an  $n$ -dimensional manifold. Then a *triangulation* of  $M$  is a complex of  $n$ -simplices on  $M$  that cover  $M$  and such that any two  $n$ -simplices are either disjoint or share at most a single  $(n-1)$ -face in common.

First, we need a lemma.

**Lemma 3.5.** *Let  $M$  be a compact  $n$ -dimensional manifold, and let  $\Delta$  be any triangulation of  $M$ . Then the  $n$ -simplices of  $\Delta$  can be arranged into a sequence  $\Delta_n^1, \Delta_n^2, \dots, \Delta_n^r$  such that, for every  $i < r$ , there is an  $(n-1)$ -face of  $\Delta_n^i$  that does not belong to any  $\Delta_n^j$ ,  $j < i$ .*

*Proof.* First set  $r$  to be the number of  $n$ -simplices in  $\Delta$ , so  $r < \infty$  because  $\Delta$  is a triangulation. Now let  $\Delta_n^r$  be any  $n$ -simplex in  $\Delta$ . Choose  $\Delta_n^{r-1}$  so that  $\Delta_n^r$  and  $\Delta_n^{r-1}$  share a common  $(n-1)$ -face. Then, because  $\Delta$  is a triangulation, it follows that any other  $n$ -simplex in  $\Delta$  cannot share that same  $(n-1)$ -face.

Now suppose  $\Delta_n^r, \Delta_n^{r-1}, \dots, \Delta_n^{i+1}$  have been chosen. Consider the complex  $K = \Delta_n^r + \Delta_n^{r-1} + \dots + \Delta_n^{i+1}$ . Again, by the fact that  $\Delta$  is a triangulation,  $K$  and its complement share a single  $(n-1)$ -face. Let  $\Delta_n^i$  be the  $n$ -simplex in the complement of  $K$  that contains that face. Then preceding by induction, the lemma is done.  $\square$

Now I will show that you can construct a field of line elements on any manifold that has a single zero.

**Theorem 3.6.** *Let  $M$  be a compact manifold of dimension  $n$ . Then there exists a smooth field of line elements on  $M$  with a single zero.*

*Proof.* Let  $\{(U_\alpha, \phi_\alpha)\}$  be an open covering of  $M$  by coordinate charts. Because  $M$  is compact we can find a number  $\epsilon > 0$ , the Lebesgue number, such that for any subset  $N \subset M$ , if  $\text{diam}(N) < \epsilon$ , then  $N \subset U_\alpha$  for some  $\alpha$ . Let  $\Delta$  be a triangulation of  $M$  into  $n$ -simplices with diameter  $< \epsilon$ . So if we order the  $n$ -simplices of  $\Delta$ ,  $\Delta_n^1, \Delta_n^2, \dots, \Delta_n^r$ , as in the preceding lemma, we have  $\overline{\Delta_n^i} \subset U_i$  for each  $i$ .

By induction, we will construct a smooth field of line elements on  $\Delta_n^1 + \dots + \Delta_n^{r-1}$  that has no zeroes. First, note that any smooth function from  $M$  to  $S^{n-1}$  determines a field of line elements with no zeroes. So on  $\Delta_n^1$  let the field be determined by any smooth function from  $U_1$  to  $S^{n-1}$ .

Now suppose the field has been constructed on  $\Delta_n^1 + \dots + \Delta_n^{i-1}$ ,  $i < r$ . If  $\Delta_n^i \cap \Delta_n^j = \emptyset$  for  $j < i$ , we can proceed as before and let the field be determined by any function from  $U_i$  to  $S^{n-1}$ . Otherwise, since  $\Delta_n^i$  has a side not belonging to any of the previous  $n$ -simplices (this is by the ordering given to them by the lemma), the intersection consists of a proper subcomplex  $K$  of  $\Delta_n^i$ . Since this is a proper subset, we can extend the field of line elements defined on  $K$  to all of  $\Delta_n^i$ .

Finally, let  $p_0$  be an interior point of  $\overline{\Delta_n^r}$ . Then for any point  $p \in \overline{\Delta_n^r}$ ,  $p \neq p_0$ , define the line element as follows. Let  $q$  be the point on boundary of  $\overline{\Delta_n^r}$  such that  $\phi_r(p_0)$ ,  $\phi_r(p)$ , and  $\phi_r(q)$  lie on the same line in  $\mathbb{R}^n$ . Then define the line element at  $p$  to simply be the same as the line element at  $r$ . Then the field of line elements is defined on all of  $M$  and it has a single zero at  $p_0$ .  $\square$

Finally, the main theorem of this section.

**Theorem 3.7.** *Let  $M$  be a compact manifold. Then  $M$  admits a smooth non-vanishing field of line elements if and only if  $\chi(M) = 0$ .*

*Proof.* Suppose  $M$  admits a smooth non-vanishing field of line elements  $X$ . Then  $\iota(X) = 0$  since  $X$  has no zeroes. Then by the Poincaré-Hopf Theorem, we have  $\chi(M) = \iota(X) = 0$ .

Conversely, suppose  $\chi(M) = 0$ . Let  $X$  be the field of line elements given by Theorem 3.6, so  $X$  has just one zero. By the Poincaré-Hopf Theorem, we have  $\iota(X) = \chi(M) = 0$ . This means that if  $p$  is the zero of  $X$ , the the index of  $X$  at  $p$  is zero. By moving to local coordinates, we can apply Theorem 3.2 to get a vector field, and hence a field of line elements, that has no zeroes at all, which finishes the proof.  $\square$

## 4 Summary

Theorems 2.2 and 3.7 let us conclude the main result of our paper that for compact manifolds, the existence of a smooth Lorentzian metric is equivalent to the manifold's having a vanishing Euler characteristic. The main consequences of this result are that it excludes certain manifolds from even being considered as models for spacetime.

In fact, for odd-dimensional compact manifolds, the result does not give any information because all odd-dimensional compact manifolds have a zero Euler characteristic. But fortunately, since spacetime is probably 4-dimensional, the result is useful.

For example, consider the spheres  $S^n \subset \mathbb{R}^{n+1}$ . In this case we can consider the vector field on  $S^n$  for which every vector points towards the north pole, which has zeros at the pole. In this case, the vectors diverge outwards at the south pole so the index there is  $+1$  and they converge inwards at the north pole so the index can be shown to be  $(-1)^n$ . By the Poincaré-Hopf Theorem, the Euler characteristic is then  $(+1) + (-1)^n$ . Since this is non-zero if  $n$  is even, we have that  $S^4$  cannot be equipped with a Lorentzian metric, i.e.  $S^4$  cannot be a model for spacetime. Furthermore, because it also follows that  $S^2$  cannot have a Lorentzian metric, it is impossible for spacetime to be a warped product such as  $S^2 \times_f S^2$ .

These few examples show the usefulness of the result in restricting our attention to only certain manifolds, and because the Euler characteristic can be calculated in a variety of ways, and sometimes very easily, the result can be readily applied in most cases.

## References

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