10.10.A) If $f_n \to f$ and $g_n \to g$, where $\{f_n\}$ and $\{g_n\}$ are in the subspace, then $\{f_n + g_n\}$ are in the subspace (since it is closed under addition), and $||(f_n + g_n) - (f + g)|| = ||(f_n - f) - (g_n - g)|| \leq ||(f_n - f|| + ||(g_n - g)|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ if n is large enough so that both $||(f_n - f||$ and $||(g_n - g)||$ are less than $\varepsilon/2$. Similarly, given $\varepsilon > 0$, if $\forall n \geq N \in \mathbb{N}$ one has $||f_n - f| < \frac{\varepsilon}{|\lambda|}$ for some fixed $0 \neq \lambda \in \mathbb{R}$, then $||\lambda f_n - \lambda f)|| = ||\lambda(f_n - f)|| = |\lambda|||f_n - f|| < |\lambda|\frac{\varepsilon}{|\lambda|} = \varepsilon$. Hence the closure of the space, contains the zero vector since it contains the subspace, contains with any two limit points their sum (as we just saw the sum is also a limit point), and also with any limit point all scalar multiples of that limit point (again, we just saw those multiples are themselves limit points). Hence, being closed under addition and scalar multiplication, the closure of a subspace is itself a subspace.

10.10.C) Assume h is strictly monotone. If $p(x)$ and $q(x)$ are polynomials, while $\lambda \in \mathbb{R}$, then $p(x) + q(x)$, $\lambda p(x)$ and $p(x) \cdot q(x)$ are also polynomials, and as $p(h(x)) + q(h(x)) = (p + q)(h(x))$, $\lambda p(h(x)) = (\lambda p)(h(x)$ and $p(h(x))q(h(x)) = (p \cdot q)(h(x))$, so the set of polynomials in h is an algebra. As the constant polynomial $\mathbf{1}$ is also, formally, a polynomial in h, this algebra does not vanish on $[0, 1]$. Finally, this algebra separates points since if $a, b \in [0, 1]$, and $p(x) \equiv x$, then $p(h(x)) = h(x)$, so $p(h(a)) = h(a) \neq h(b) = p(h(b))$ where the middle inequality holds because h is monotone. Thus the algebra of polynomials in h satisfies the conditions, and hence the conclusion, of the Stone-Weierstrass Theorem, i.e. it is dense in $C([0, 1])$. Therefore every $f \in C([0, 1])$ is a limit of polynomials in h.

Conversely, suppose any $f \in C([0, 1])$ is a limit of polynomials in a function h. If h is not monotone, since it is continuous, by the Intermediate Value Theorem it is not one-to-one. Hence $\exists\ x, y \in [0, 1]$ with $x \neq y$ and $h(x) = h(y)$. But then for any polynomial p, $p(h(x)) = p(h(y))$. Therefore, for any limit of f of polynomials in h, we have equality of pointwise limits at x and y, i.e. $f(x) = f(y)$. It follows that a function such as $f(x) := x$, for which $f(x) \neq f(y)$ if $x \neq y$, cannot be approximated by polynomials in h, and this is a contradiction. Thus h must be monotone.

10.10.D) Since $\overline{A} \neq C(X)$ and A separates points, we know that A vanishes on *at least* one point, since otherwise this inequality of sets would be an equality, by Stone-Weierstrass. But it really vanishes on *exactly* one point - otherwise it would not separate points: If A vanishes on $x, y \in X, x \neq y$, then for any $f \in A$, $f(x) = f(y) = 0$, so, as no f in A gives different values on x and y, A does not separate x from y. Denote then the unique point where A vanishes by x_0. Clearly $\overline{A} \subset B$, where $B := \{f \in C(X) \,|\, f(x_0) = 0\}$: by the above $A \subset B$, and if f is a limit point of a sequence of functions $f_n \in A$, it is a limit in the infinity norm, i.e. the sequence converges uniformally to f, in particular it converges pointwise, so $0 = f_n(x_0) \to f(x_0)$ as $n \to \infty$, so $f(x_0) = 0$ (as a limit of a constant sequence consisting only of zeros). To show the converse inclusion, we proceed as follows. $A + \mathbb{R}(1)$ (i.e. $A+$constants) is an algbera (e.g., for constants, c, d, and $f, g \in A$, we have $(f + c)(g + d) = (fg + df + cg) + cd$, and the term is bracket is in A since A is an algebra, while cd is a constant. Hence $(f + c)(g + d) \in A + \mathbb{R}(1)$). Moreover, this algebra separates points (as A does and $A = A+0 \subset A+\mathbb{R}1$, so for any $x, y \in X$ there is an element $f + 0$ that has distinct values on x and y). Finally, $A+\mathbb{R}(1)$ is nowhere vanishing, since the function whose value is identically 1 belongs to this algebra (as $0 + 1$, where 0 is the zero function, which belongs to A since A is an algebra, hence in particular a vector subspace). By Stone-Weierstrass, $\overline{A + \mathbb{R}1} = C(X)$. Clearly $B + \mathbb{R}1 \subset C(X)$, and $C(X) \subset B + \mathbb{R}1$ since any continuous function g on X satisfies $g = (g - g(x_0) \cdot 1) + g(x_0) \cdot 1$, and the term in brackets belongs to B. Therefore $B+\mathbb{R}1 = C(X)$. The last two set equalities give $\overline{A + \mathbb{R}1} = B+\mathbb{R}1$. Suppose then that $f \in B$. If c is a constant, the last equality of sets guarantees the existence of $f_n \in A$ and constants c_n such that $f_n + c_n \cdot 1 \to f + c \cdot 1$ uniformally on X. Then at x_0, where f_n's and f vanish, we get $c_n \to c$. Therefore, $\lim f_n = \lim(f+(c_n-c)\cdot 1) = f+0\cdot 1 = f$. Thus, $B \subset \overline{A}$, and since we have already shown the opposite inclusion, we have $\overline{A} = B$.

9.3.B) Consider the Euclidean metric on $\mathbb{R}^2$. Any line L is a closed set in $\mathbb{R}^2$, since limit points of sequences of points in L are still in the line, so L contains all its limit points. The interior of L is empty, since for any $p \in L$ and $r > 0$, $B_r(p)$ is a disk, and therefore it contains points not in L (Thus no $p \in L$ is contained in an open set which is contained in L, so $\mathrm{Int}(L)$ must be empty). By Baire's theorem, since $\mathbb{R}^2$ is complete, a countable union of lines must have an empty interior, so it cannot be all of $\mathbb{R}^2$, whose interior is itself and thus is not empty.

9.3.C) If $X = \cup_{k=1}^{\infty}\{x_k\}$, and none of the $\{x_k\}$ are open sets, then, since each $\{x_k\}$ is closed (it contains its only limit point x_k), its closure is itself and thus has empty interior. Hence, by Baire's Theorem, X has empty interior, but the whole space is always an open set, and therefore $Int(X) = X \neq \emptyset$. This contradiction means that X has sets of the form $\{x_k\}$ which are open sets - i.e. isolated points.

Question 3) $\partial A = \overline{A} \setminus \mathrm{Int} A$. Hence, one can check the set theoretic equality $X \setminus \partial A = X \setminus (\overline{A} \setminus \mathrm{Int}(A)) = \mathrm{Int}(A) \cup (X \setminus \overline{A})$. The set on the right is a union of open sets, so open. Hence ∂A is a closed set. Thus $\mathrm{Int}(\overline{\partial A}) = \mathrm{Int}(\partial A)$, and the question is whether or not the latter set is empty. Now suppose that A is open. Then, if $p \in \mathrm{Int}(\partial A) \subset \partial A \subset \overline{A}$, it follows that p is a limit point of points in A. Openness of $\mathrm{Int}(\partial A)$ means that there exists a ball $B_r(p) \subset \mathrm{Int}(\partial A) \subset \partial A$, which must contain points of A (again since p is a limit point). Thus $\emptyset \neq B_r(p) \cap A \subset \partial A \cap A$, but this is a contradiction, since $\partial A = \overline{A} \setminus A$ cannot intersect A. Therfore no such p exists, $\mathrm{Int}(\overline{\partial A})$ is empty and ∂A is nowhere dense.

If A is not open, this argument does not work, since then $\mathrm{Int}(A)$ will not equal A, so the intersection $\partial A \cap A$ need not be empty, and no contradiction arises. For example, if $A = \mathbb{Q}$ in the space $X = \mathbb{R}$, we know that $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathrm{Int}(\mathbb{Q}) = \emptyset$, so $\partial \mathbb{Q} = \mathbb{R} \setminus \emptyset = \mathbb{R}$, hence $\mathrm{Int}(\partial \mathbb{Q}) = \mathbb{R} \neq \emptyset$, so $\partial \mathbb{Q}$ is *not* nowhere dense.