8.1 K. (a) Uniform continuity means $\forall\varepsilon > 0\ \exists\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$. Now, given $\varepsilon > 0$, let $N \in \mathbb{N}$ satisfy $\frac{1}{N} < \delta$ for $\delta = \delta(\varepsilon)$ as above. Then, if $n \geq N$, we have, for every $x \in \mathbb{R}$, that $|f_n(x) - f(x)| = |f(x + \frac{1}{n}) - f(x)| < \varepsilon$, since $|x + \frac{1}{n} - x| = \frac{1}{n} \leq \frac{1}{N} < \delta$ (and we are thinking as $x + \frac{1}{n}$ as some y in the uniform continuity definition).

(b) No. Take $f(x) = x^2$ on $\mathbb{R}$. Then $|f_n(x) - f(x)| = |f(x + \frac{1}{n}) - f(x)| = |(x + \frac{1}{n})^2 - x^2| = |2\frac{x}{n} + \frac{1}{n^2}|$. Now take $\varepsilon = 1$. For any integer N, taking $x = N$ and $n = N$, we see that $|f_N(N) - f(N)| = 2 + \frac{1}{N^2} > 1 = \varepsilon$.

8.2 B. First, as $(n + x)/4n + x) = (1 + x/n)/(4 + x/n) \to 1/4$ as $n \to \infty$ for any $x \in \mathbb{R}$, we see that $h_n(x)$ converges *pointwise* to $h(x) \equiv \frac{1}{4}$. Now $|h_n(x) - h(x)| = \frac{3x}{4(4n+x)}$. If $x \in [0, N]$, the last quantity is $< 4x/(4(4n + x) = x/(4n + x)) \leq \frac{x}{4n} \leq \frac{N}{4n}$. So, given $\varepsilon > 0$, let $K \in \mathbb{N}$ with $K > \frac{N}{4\varepsilon}$. We then have $|h_n(x) - h(x)| < \varepsilon$ for every $x \in [0, N]$ and all $n \geq K$.

On the other hand, given $\varepsilon = 1/10$, for any integer $K \in \mathbb{N}$, if $n > K$ and $x = n$, we have $|h_n(n) - h(n)| = \frac{3n}{20n} = \frac{3}{20} > \frac{1}{10} = \varepsilon$, contradicting uniform convergence on $[0, \infty)$.

8.2 C. This is a simple consequence from the fact that if f_n converges uniformally to f on a compact set, then f is continuous on that set. In the case at hand, as $[a, b]$ is compact, the restriction $f|_{[a,b]}$ is continuous. But continuity of f on $(0, 1)$ simply means continuity of f at every $x \in (0, 1)$. Hence, given such x, there exists a and b with $0 < a < x < b < 1$. But f is continuous on $[a, b]$, so in particular it is continuous at $x \in [a, b]$. Thus, f is indeed continous at every $x \in (0, 1)$.

8.3 F. The integral defining J_0, although improper, is well-defined. One can check that in these circumstances, differentiation under the integral sign still satisfies Leibniz's Rule 8.3.4, essentially with the same proof. Given that, we compute $J_0'(x) = \frac{1}{\pi}\int_{-1}^{1}\left(\frac{-t}{\sqrt{1-t^2}}\sin(xt)\right)dt$, and $J_0''(x) = \frac{1}{\pi}\int_{-1}^{1}\left(\frac{-t^2}{\sqrt{1-t^2}}\cos(xt)\right)dt$. Thus, $J_0''(x) + J_0'(x)/x + J_0(x) = \frac{1}{\pi}\int_{-1}^{1}\left[\frac{-t^2+1}{\sqrt{1-t^2}}\cos(xt) + \frac{-t}{\sqrt{1-t^2}}\frac{\sin(xt)}{x}\right]dt = \frac{1}{\pi}\int_{-1}^{1}\left[\sqrt{1-t^2}\cos(xt) + \frac{-t}{\sqrt{1-t^2}}\frac{\sin(xt)}{x}\right]dt = \frac{1}{\pi}\int_{-1}^{1}\left[\sqrt{1-t^2}\frac{d}{dt}\left(\frac{\sin(xt)}{x}\right) + \frac{d}{dt}\left(\sqrt{1-t^2}\right)\frac{\sin(xt)}{x}\right]dt = \frac{1}{\pi}\int_{-1}^{1}\frac{d}{dt}\left[\sqrt{1-t^2}\frac{\sin(xt)}{x}\right]dt = \frac{1}{\pi}\left[\sqrt{1-t^2}\frac{\sin(xt)}{x}\right]\Big|_{-1}^{1} = 0 - 0 = 0$.

8.4 D. Yes: $\forall x \in \mathbb{R}$, $\frac{1}{x^2+n^2} \leq \frac{1}{n^2}$, and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (by the Calculus I p-test, with $p = 2 > 1$). Hence, the series converges uniformally on $\mathbb{R}$ by the Weierstrass M-test.

8.4. H. We are given that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N, |s_n(x) - s(x)| < \varepsilon/2$ for every $x \in [0, 1]$ and some function $s(x)$ on $[0, 1]$. Hence, given $\varepsilon > 0$, take N as above. We have, for any $n \geq N$, that $|a_n(x)| = |s_{n+1}(x) - s_n(x)| = |s_{n+1}(x) - s(x) + s(x) - s_n(x)| \leq |s_{n+1}(x) - s(x)| + |s(x) - s_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, for every $x \in [0, 1]$. Therefore $a_k(x)$ converge uniformally to 0 (on $[0, 1]$).