7.5 A. Equality holds when the vectors $\{e_1, e_2, \dots, e_n\}$ span the entire space. That is, when they are an orthonormal basis of V.

7.5 E. Let P and Q be orthogonal projections on the subspace M. For all $x \in V$ we have $x = Px + (I - P)x$. Here $Px \in M$ and $(I - P)x \in M^\perp$. Multiplying by Q we get $Qx = QPx + Q(I - P)x$. Since $\mathrm{Range}(Q) = M$ we have $QPx = Px$. Since $\ker Q = M^\perp$ we have $Q(I - P)x = 0$. So, we conclude that $Qx = Px$ for all $x \in V$.

7.5. G. (a) We will use the following two facts. Let A, B be subspaces (not necessarily closed) of a Hilbert space H. Then we have: (1) if $A \subset B$ then $B^\perp \subset A^\perp$, (2) $(A^\perp)^\perp = \overline{A}$.

Let M, N be as in the statement of the exercise. Then $M \cap N \subset M$, hence $M^\perp \subset (M \cap N)^\perp$. In the same way we get that $N^\perp \subset (M \cap N)^\perp$. Since $(M \cap N)^\perp$ is a closed linear subspace, we conclude that $\overline{M^\perp + N^\perp} \subset (M \cap N)^\perp$.

We have $M^\perp \subset \overline{M^\perp + N^\perp}$, hence $(\overline{M^\perp + N^\perp})^\perp \subset M$. In the same way we have $(\overline{M^\perp + N^\perp})^\perp \subset N$. Hence, $(\overline{M^\perp + N^\perp})^\perp \subset (M \cap N)$. Thus we conclude that $(M \cap N)^\perp \subset \overline{M^\perp + N^\perp}$.

(b) Let M, N be as in the statement of the exercise (in part (b)). We notice that $e_{2n} \in M \subset N + M$ and $e_{2n-1} = (e_{2n-1} + ne_{2n}) + (-ne_{2n}) \in N + M$. Since $M + N$ contains the orthogonal basis $\{e_n\}_{n=1}^\infty$ it must be a dense subspace. In order to prove that it is not closed it is enough to find a vector of l_2 that is not in $M + N$. Let $z = \sum \frac{1}{n} e_{2n-1}$ and suppose that $z = u + v$, where $u \in N$ and $v \in M$. We notice that $\langle x, e_{2n-1} \rangle = 0$ for all $x \in M$. In order to prove this it is enough to check that it is true for the generators of M (and $\langle e_{2m}, e_{2n-1} \rangle = 0$ is clearly true). Thus we have

$$\frac{1}{n} = \langle z, e_{2n-1} \rangle = \langle u, e_{2n-1} \rangle + \langle v, e_{2n-1} \rangle = \langle u, e_{2n-1} \rangle.$$

We also have $\langle x, e_{2n-1} \rangle = \frac{1}{n} \langle x, e_{2n} \rangle$ for all $x \in N$ (as before, it is enough to check this for the generators). Thus we conclude that $\frac{1}{n} = \langle z, e_{2n-1} \rangle = \langle u, e_{2n-1} \rangle = \frac{1}{n} \langle u, e_{2n} \rangle$. We get that $\langle u, e_{2n} \rangle = 1$ for all $n \geq 1$. But this is impossible because the numbers $\langle u, e_{2n} \rangle$ must form a square summable series. Hence there are no $u \in N$, $v \in M$ such that $z = u + v$; that is, $z \notin N + M$.

(c) In part (b) we found closed subspaces M and N such that $M + N$ is not closed. Let $M_1 = M^\perp$ and $N_1 = N^\perp$. If we let $M = M_1$ and $N = N_1$ in the equality of part (a) we get an example where the equality does not hold if one doesn't take closure on the right side.

7.6 E. The set P_n of all polynomials of degree at most n is a linear subspace of $C[a, b]$. P_n is finite dimensional since it is spanned by the basis $\{1, x, \dots, x^n\}$. Therefore, by Theorem 7.6.5, for every $f \in C[a, b]$ there is a closest point in P_n with respect to the norm $\| \cdot \|_\infty$.

7.6. G. Let w_1 and w_2 be two points of W that are closest to $v \in V$. This means that $\|v - w_1\| = \|v - w_2\| \leq \|v - w\|$ for all $w \in W$. Let $w_3 = \frac{w_1 + w_2}{2}$. Then $w_3 \in W$ and

$$\|v - w_3\| = \|\frac{v - w_1}{2} + \frac{v - w_2}{2}\| \leq \|\frac{v - w_1}{2}\| + \|\frac{v - w_2}{2}\| = \|v - w_1\|$$

Thus, we have $\|v - w_3\| \leq \|v - w_1\|$. We also have $\|v - w_1\| \leq \|v - w_3\|$ (by the property of w_1 of being a closest point to v). Hence we conclude that $\|\frac{(v-w_1)+(v-w_2)}{2}\| = \|v - w_1\| = \|v - w_2\|$. By the strict convexity of the space, this implies that $v - w_1 = v - w_2$; hence $w_1 = w_2$. So the closest point to v must be unique.

5.1. I. The mean value theorem says that for all $x, y \in [a, b]$ there is $\eta \in [x, y]$ such that $f(x) - f(y) = f'(\eta)(x - y)$. Since we are assuming that $|f'(\eta) \leq M$ we get that $|f(x) - f(y)| \leq M|x - y|$.

5.4 A If A is not a compact subset of $\mathbb{R}^n$ then it fails to be closed or bounded. Suppose that A is not closed. Let x_0 be a limit point of A such that $x_0 \notin A$. Then the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = -\min(\|x - x_0\|, 1)$ is bounded between -1 and 0 and has supremum equal to 0, but does not attain the value 0. Suppose that A is unbounded. Then the function $f(x) = -\frac{1}{\max(\|x\|,1)}$ is bounded between -1 and 0 and has supremum equal to 0, but does not attain the value 0.