

7.1 A. First notice that  $\|(x, y, z)\| = |x| + 2\|(y, z)\|_2$ , where  $\|(y, z)\|_2$  is the Euclidean norm in  $\mathbb{R}^2$ . If  $|x| + 2\|(y, z)\|_2 = 0$  then  $|x| = 0$  and  $\|(y, z)\|_2 = 0$ ; hence  $(x, y, z) = (0, 0, 0)$ . Also  $\|\alpha(x, y, z)\| = |\alpha x| + \|\alpha(y, z)\|_2 = |\alpha|(|x| + \|(y, z)\|_2) = \|(x, y, z)\|$ . Finally, we check that  $\|\cdot\|$  satisfies the triangle inequality.

$$\begin{aligned} \|(x_1, y_1, z_1) + (x_2, y_2, z_2)\| &= |x_1 + x_2| + 2\|(y_1, z_1) + (y_2, z_2)\|_2 \\ &\leq |x_1| + |x_2| + 2(\|(y_1, z_1)\|_2 + \|(y_2, z_2)\|_2) \\ &= \|(x_1, y_1, z_1)\| + \|(x_2, y_2, z_2)\|. \end{aligned}$$

7.1. C.  $\rho(f) = \|f'\|_\infty$  is the  $\|\cdot\|_\infty$  norm of  $f'$ ; hence it is nonnegative. We have  $\rho(\alpha f) = \|(\alpha f)'\|_\infty = |\alpha|\|f'\|_\infty = |\alpha|\rho(f)$ . Also,  $\rho(f + g) = \|(f + g)'\|_\infty = \|f' + g'\|_\infty \leq \|f'\|_\infty + \|g'\|_\infty = \rho(f) + \rho(g)$ . This function is not a norm because  $\rho(1) = 0$  but  $1 \neq 0$ .

7.2. D. Since  $U$  is open and translation by a vector is a continuous function we know that the set  $v + U$  is open for all  $v \in V$ . We have  $A + U = \bigcup_{a \in A} a + U$ . Hence  $A + U$  is open, since it is a union of open sets.

7.3. C. Let  $x \neq 0$  be a nonzero vector. Then  $0 = \|x + (-x)\| < \|x\| + \|-x\| = 2\|x\|$ . Thus,  $x$  and  $-x$  are collinear, but the triangle inequality between them is a strict inequality.

7.3 F. (a) Suppose that  $u, v$  satisfy that  $2\|u\| = 2\|v\| = \|u + v\|$ . By the parallelogram law we have  $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$ . The conclusion then follows by substituting in the first term on the left hand side as follows:  $\|u + v\|^2 = 4\|u\|^2 = 2\|u\|^2 + 2\|u\|^2 = 2\|u\|^2 + 2\|v\|^2$ . Cancelling on both sides we get that  $\|u - v\| = 0$ . Thus we have  $u = v$ .

(b) Let  $u = (1, 1)$  and  $v = (1, -1)$ . Then  $\|u\|_\infty = \|v\|_\infty = 1$  and  $\|u + v\|_\infty = 2$ , but  $u \neq v$ .

7.4 A. See the solution of Problem 6 in Assignment 1.

7.4 G. We will use the identity  $2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$ . We have

$$\begin{aligned} \int_0^1 \sin(n\pi x) \sin(m\pi x) dx &= \frac{1}{2} \int_0^1 \cos((n - m)\pi x) - \cos((n + m)\pi x) dx \\ &= \frac{1}{2} \delta_{n,m}, \end{aligned}$$

where  $\delta_{n,m} = 0$  if  $n \neq m$  and  $\delta_{n,n} = 1$ . So strictly, these functions are not orthonormal, only orthogonal.