4.3 A (a) By the triangle inequality we have

$$0 \leq |\|x_n\| - \|a\|| \leq \|x_n - a\|.$$

Since $\|x_n - a\| \to 0$ when $n \to \infty$ we conclude that $\|x_n\| \to \|a\|$ when $n \to \infty$.

(b) Let $x_n = 1$ for all $n \in \mathbb{N}$ and $a = -1$.

9.1 B. Let X be a discrete metric space. We have $B_1(a) = \{a\}$ for all $a \in X$; that is, the open ball with center a and radius 1 is equal to the singleton set $\{a\}$. Thus, the set $\{a\}$ is open. Since every set is a union of singleton sets and the union of open sets is open every subset of X is an open set. Every subset of X is the complement of its complement; thus it is the complement of some open subset. Hence every subset of X is also closed.

9.2 M. (a) Let us define the function $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by

$$d(n,m) = \begin{cases} \frac{1}{2} + \frac{1}{2n} & n > m \\ 0 & n = m \\ d(m,n) & n < m \end{cases}$$

Let us prove that d is a metric. It is clear that by definition $d(n,m) = d(m,n)$ and $d(n,m) = 0$ if and only if $n = m$. Also notice that if $n \neq m$ then $1/2 < d(n,m) \leq 1$. So if n, m, k are three different natural numbers then

$$d(n,m) \leq 1 = \frac{1}{2} + \frac{1}{2} \leq d(n,k) + d(k,m).$$

It is also easy to check the triangle inequality when some of the numbers n, m, k are equal (in this case the triangle inequality becomes equality). So $d(\cdot,\cdot)$ is a metric on $\mathbb{N}$. Since $B^d_{\frac{1}{2}}(n) = \{n\}$ for every n the Cauchy sequences are eventually constant (see exercise 4). Thus $(\mathbb{N}, d)$ is complete. Finally, one can check that

$$\overline{B^d_{\frac{1}{2}+\frac{1}{2n+1}}(n)} = \{n, n+1, \dots\}.$$

Hence the balls with center n and radius $\frac{1}{2} + \frac{1}{2n+1}$ form a nested sequence of closed balls with empty intersection.

(b) Suppose that (M, d) is complete. Let $\overline{B_{r_n}(a_n)}$ be a sequence of nested closed balls. Let us prove that $\{a_n\}_{m=1}^{\infty}$ (the sequence of centers) is a Cauchy sequence. For every ϵ there is N such that $r_N < \epsilon$ (because $r_n \to 0$). Then for every $n > N$ the points a_n are all inside the ball $\overline{B_{r_N}(a_N)}$. We have

$$d(a_n, a_m) \leq d(a_n, a_N) + d(a_m, a_N) \leq 2r_N < 2\epsilon.$$

Hence the sequence $\{a_n\}_{n=0}^{\infty}$ is Cauchy. Since M is complete this sequence must be convergent to some $a \in M$. One can prove that this point belongs to the intersection of all the balls.

Suppose now that (M, d) has the property that every sequence of nested closed balls has nonempty intersection. We want to prove that (M, d) is complete. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M. Then for every k there is n_k such that $d(x_n, x_m) < \frac{1}{2^{k+1}}$ for all $n, m \geq n_k$. Now consider the sequence of closed balls $B_k = \overline{B_{\frac{1}{2^k}}(x_{n_k})}$. Let us prove that this is a sequence of nested closed balls. Let $x \in B_{k+1}$. Then $d(x, x_{n_k}) \leq d(x, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}) \leq \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}$. Thus, we get that $x \in B_k$. Since $\{B_k\}_{k=1}^{\infty}$ forms a sequence of nested closed ball, they have nonempty intersection. Let $a \in \cap_{k=1}^{\infty} B_k$. Then $d(a, x_n) < \frac{1}{2^k}$ for all $n \geq n_k$. This implies that $\lim_n x_n = a$.

3. Let $\{B_1(x_i)\}_{i=1}^n$ be a finite covering of the space with balls of radius 1 (this is always possible since A is totally bounded). Let $r = \max_i d(x_i, x_1)$. Let us prove that $A = B_{r+1}(x_1)$. Let $x \in A$ be some point in A. Then $x \in B_1(x_i)$ for some i. So $d(x, x_1) \leq d(x, x_i) + d(x_i, x_1) \leq 1 + r$. This proves our claim.

4. Let $a \in \mathbb{N}$. We notice that $B_1^{\rho}(a) = \{a\}$. Hence for all $r > 0$ we have $B_1^{\rho}(a) = \{a\} \subset B_r^d(a)$. Let $s = \min(\frac{1}{a} - \frac{1}{a+1}, \frac{1}{a-1} - \frac{1}{a})$. Then $B_s^d(a) = \{a\}$. Hence for all $r > 0$ we have $B_s^d(a) = \{a\} \subset B_r^{\rho}(a)$. Thus the two metrics have the same open sets and the same convergent subsequences by Problem 9.1.G.

Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathbb{N}, \rho)$. Since $|a_n - a_m| < 1$ for all $n, m > N$ for some N we conclude that $a_n = a_m$ for all $n, m > N$. That is, the sequence is eventually constant. Hence it is convergent. This proves that $(\mathbb{N}, \rho)$ is complete. The sequence $a_n = n$ is Cauchy in $(\mathbb{N}, d)$ but is not convergent: If $a \in \mathbb{N}$ is the limit, we just take ε to be s above, and then for any $n > a$, $d(n, a) \geq s = \varepsilon$. Hence $(\mathbb{N}, d)$ is not complete.