

1) J)a) We have $\rho_2(a, a) = 0$ by definition. If $a \neq b$ then $a - b = 2^e \frac{f}{g}$ for some integer e , thus $\rho_2(a, b) \neq 0$.

If $a - b = 2^e \frac{f}{g}$ then $b - a = 2^e \frac{(-f)}{g}$; thus $\rho_2(a, b) = \rho_2(b, a)$.

Finally we prove the triangle inequality. Let $a, b, c \in \mathbb{Q}$ be such that $a - b = 2^e \frac{f}{g}$ and $b - c = 2^{e'} \frac{f'}{g'}$. Then

$$a - c = 2^e \frac{f}{g} + 2^{e'} \frac{f'}{g'} = 2^{\min(e, e')} \left(2^{e - \min(e, e')} \frac{f}{g} + 2^{e' - \min(e, e')} \frac{f'}{g'} \right).$$

Thus $\rho_2(a, c) \leq \min(\rho_2(a, b), \rho_2(a, c)) \leq \rho_2(a, b) + \rho_2(a, c)$.

b) We have $a_n - \frac{1}{3} = -\frac{(-2)^n}{3}$. From this we get that $\rho_2(a_n, \frac{1}{3}) = 2^{-n}$ which tends to 0 as $n \rightarrow \infty$. Thus the sequence converges to $\frac{1}{3}$.

c) We have that $n! + 1$ is odd and $n! = 1 \cdot 2 \cdot 3 \cdots n = 2^{\lfloor n/2 \rfloor} k$, for some $k \in \mathbb{N}$. Thus $\frac{n!}{n!+1} = 2^{\lfloor n/2 \rfloor} \frac{k}{m}$ with m odd. This implies that $\rho_2(\frac{n!}{n!+1}, 0) \rightarrow 0$ when $n \rightarrow \infty$. Hence the sequence converges to 0.

2) A(c) The closure of $\{(x, \sin(\frac{1}{x})) : x > 0\}$ is

$$\{(x, \sin(\frac{1}{x})) : x > 0\} \cup \{(0, y) : y \in [-1, 1]\}.$$

J. Since A is closed A^c is open. We have $U \setminus A = U \cap A^c$. Thus $U \setminus A$ is open since it is the intersection of two open sets. In the same way $A \setminus U = A \cap U^c$ is the intersection of two closed sets; thus it is a closed set.

L(a) Let $r \in \mathbb{Q}$. Then $r + \sqrt{2}/n$ is a sequence of irrational numbers that tends to r when $n \rightarrow \infty$. Thus the closure of the irrational numbers contains \mathbb{Q} . Since \mathbb{Q} is dense in \mathbb{R} we conclude that $\mathbb{R} \setminus \mathbb{Q}$ is dense in \mathbb{R} .

3) a) We will show that A^c is open. Let $x \in A$ and $d = \rho(x, a) > r$. Let $d' = d - r$. Let us prove that the open ball $B_{d'}(x)$ with center x and radius d' has empty intersection with A . If $a' \in A$ then $\rho(a', x) \geq \rho(x, a) - \rho(a', a) \geq d - r = d'$. Thus $a' \notin B_{d'}(x)$. Since we can do this for every $x \in A^c$ we conclude that A^c is open.

Let X be a metric space such that $\rho(x, y) = 1$ if $x \neq y$ and $\rho(x, x) = 0$ (a discrete metric space). Then $\overline{B_1(a)} = a$ and $A = X$. So if X contains more than one point we get $A \neq \overline{B_1(a)}$.

b) We will show that A^c is open. Let $x \in A^c$ and $d = \rho(x, a) > 0$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a there is N such that $\rho(x_n, a) < d/2$ for all $n > N$. Let us define $d' = \min\{\rho(x_1, x), \dots, \rho(x_N, x), d/2\}$. Since d' is the minimum of a finite set of positive numbers we have that $d' > 0$. Let us prove that $B_{d'}(x)$ doesn't intersect A . For $\{x_n\}_{n=1}^N$ we have $\rho(x_n, x) \geq d'$. If $n > N$ then $\rho(x_n, x) \geq \rho(x, a) - \rho(x_n, a) \geq d - d/2 \geq d/2 \geq d'$. Since we can find an open ball around every point of A^c that doesn't intersect A we conclude that A^c is open.

4) b) The closure of A is the intersection of all the closed subsets that contain A . Since $A \subset B \subset \overline{B}$ (i.e. A is contained in \overline{B}) we must have $\overline{A} \subset \overline{B}$.

The converse does not hold. Let $B = (0, 1)$ and $A = [0, 1]$. Then $\overline{A} \subset \overline{B}$ (they are equal) but A is not a subset of B .

a) In order to prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ we prove that the right side is contained in the left side and viceversa. We have $A \cup B \subset \overline{A} \cup \overline{B}$. Thus $\overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. This proves one inclusion. We have $A \subset A \cup B$ and $B \subset A \cup B$. Thus $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$. This implies that $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

We have $A \cap B \subset A$ and $A \cap B \subset B$. Taking closure on both sides of these inclusions we get $\overline{A \cap B} \subset \overline{A}$ and $\overline{A \cap B} \subset \overline{B}$. From this we get $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. This inclusion can be proper. Example: $A = (0, 1)$ and $B = (1, 2)$.