MAT 198 - Completion of a practice problem

For which k does the following system of equations

$$\begin{aligned} 2x + ky - z &= 0 \\ y + kz &= 2 \\ kx + y &= 2 \end{aligned}$$

have
(a) no solutions?
(b) a unique solution?
(c) infinitely many solutions?

Answer: The coefficient matrix of this system is

$$A = \begin{bmatrix} 2 & k & -1 \\ 0 & 1 & k \\ k & 1 & 0 \end{bmatrix}.$$

As computed in class, its determinant is $k(k-1)(k+1)$. If $k \neq 0$, 1 or -1, this determinant is not zero, A is invertible and hence there is a unique solution – see for example (a) and (f) of Theorem 7.1.5 on page 344 of the book.

If k is one of the above three values there can be solutions. Part (e) of the theorem above only says that in that case $A\mathbf{x} = \mathbf{b}$ is not consistent for *every* $\mathbf{b}$, but it can be consistent for *some* $\mathbf{b}$'s, such as the column vector whose components are all 2 as in our case. We do know by part (f) that there are either more or less than exactly one solution. There is a quick way to arrive at the answer, but this will be given in the end - first we examine these cases slowly.

If $k = 0$ the matrix has its last two rows coincide, so there are two distinct row vectors. They are not multiples of each other, hence they are linearly independent. So row space is two-dimensional, hence so is column space (Theorem 5.6.1), so rank$(A) = 2$, hence nullity$(A) = 1$ (Theorem 5.6.3). So the number of parameters in the homogeneous system is 1, and as the general solution of the inhomogeneous system is a particular solution plus the general solution to the homogeneous system, there is also a 1 parameter family of those, hence the system has infinitely many solutions when $k = 0$.

When $k = 1$ one can investigate similarly, or notice that

$$b = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

is a multiple of the second column vector, hence it is in column space. By the Consistency Theorem 5.6.5 $Ax = b$ is consistent. Since we already know it cannot have exactly one solution (as A is not invertible), it has more. By Theorem 1.6.1 it has infinitely many. One can find more precise information by computing its rank and from that the number of parameters.

Finally, if $k = -1$, one can check that A has reduced row-echelon form given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This has two leading ones, hence rank two, so nullity one, so one parameter family of solutions to homogeneous and then inhomogeneous – so there are infinitely many solutions here too.

Tue quick way is too notice by (j) of Theorem 5.6.5 that since A is not invertible, its columns are linearly dependent, so the rank must be less than $n = 3$, hence the nullity must be more than 0, hence there are a finite number of parameters in the general solution, hence infinitely many solutions.