A non-chainable continuum with span zero

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Chainability

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Fact

If X is chainable, C is a continuum, $f, g : C \to X$ are continuous with $f(C) = g(C)$, then there is some $x \in C$ with $f(x) = g(x)$.

Span

A. Lelek (1964) considered the quantity

$$\sup_{C,f,g} \inf_{x \in C} d(f(x), g(x))$$

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This is equal to:

Definition (Lelek, 1964)

$$\mathrm{Span}(X) = \sup_{Z} \inf_{(x,y) \in Z} d(x, y)$$

where Z ranges over all continua $Z \subset X \times X$ with $\pi_1(Z) = \pi_2(Z)$ (here $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$).

Lelek's Problem

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If X is chainable then $\mathrm{Span}(X) = 0$.

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Question (Lelek, 1971)

Is it true that if $\mathrm{Span}(X) = 0$ then X is chainable?

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Fact (Bing, 1951)

The pseudo-arc is the only hereditarily indecomposable chainable continuum.

Span example: Circle

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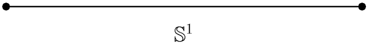

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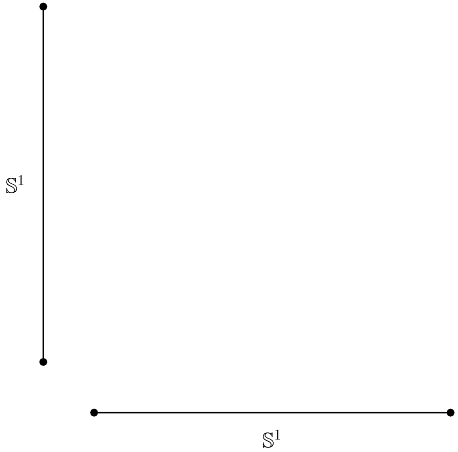

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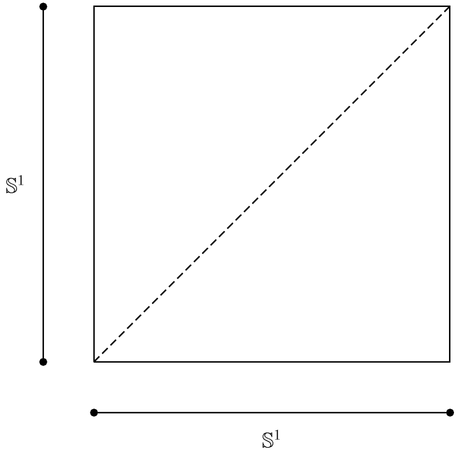

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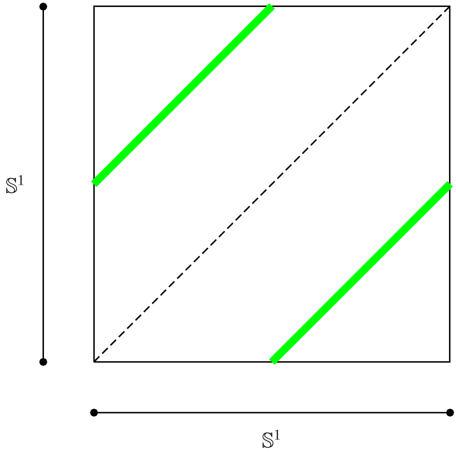

Define $Z \subset \mathbb{S}^1 \times \mathbb{S}^1$ as shown. This witnesses that $\mathrm{Span}(\mathbb{S}^1) > 0$.

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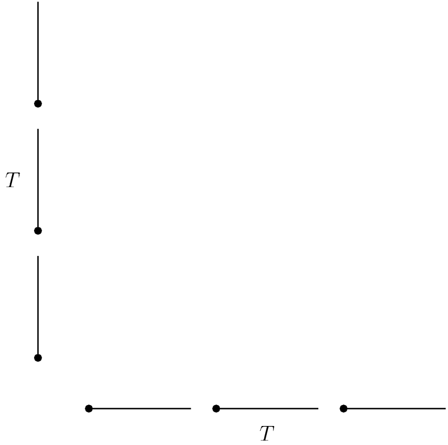

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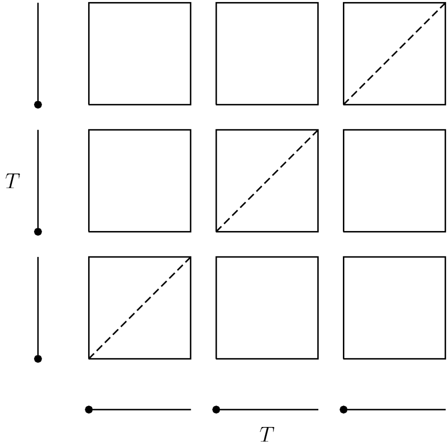

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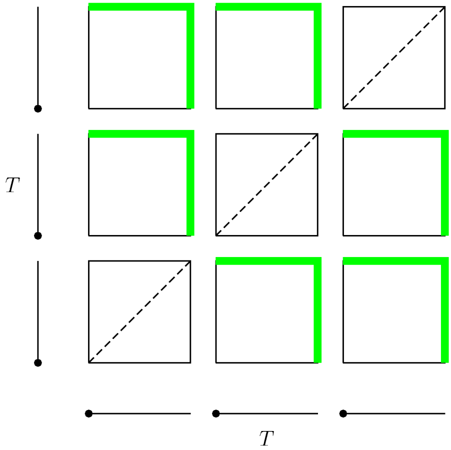

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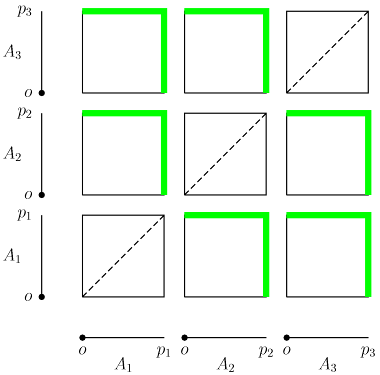
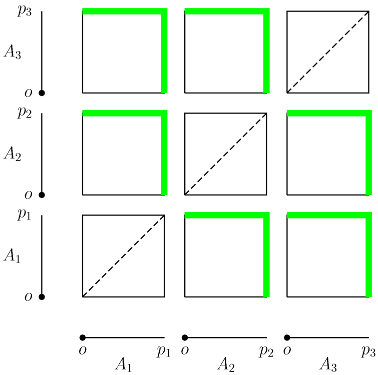
Simple triods with small span



Simple triods with small span

Proposition

Suppose $\delta > 0$ and $W \subset A_1 \times A_2$ is an arc such that $(o, o) \in W$, W meets $(\{p_1\} \times A_2) \cup (A_1 \times \{p_2\})$, and $d(x, y) \leq \delta$ for each $(x, y) \in W$. Then $\mathrm{Span}(T) \leq \delta$.



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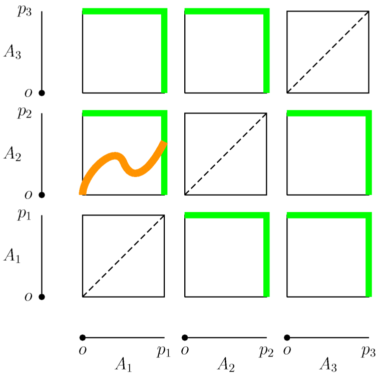

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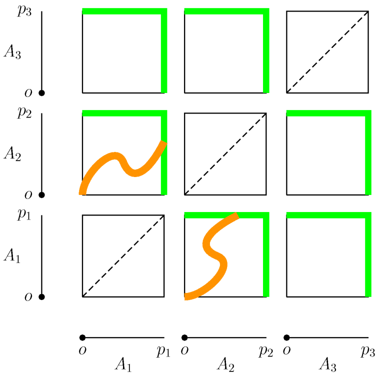

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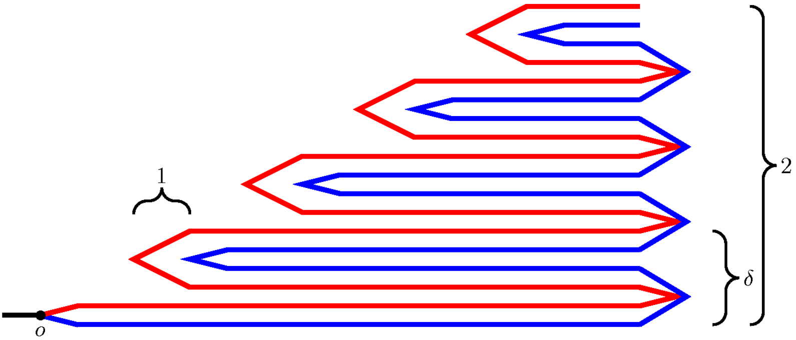

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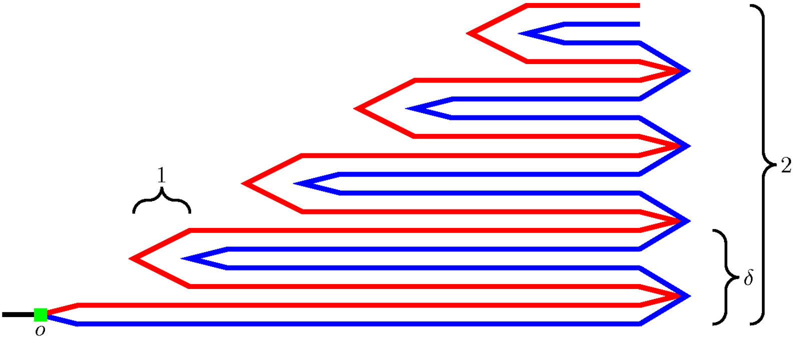

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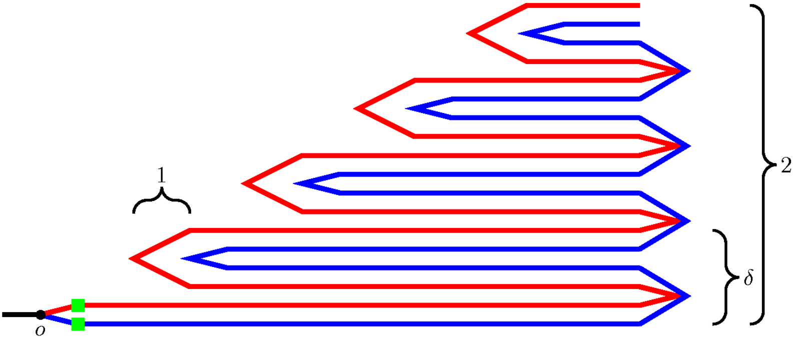

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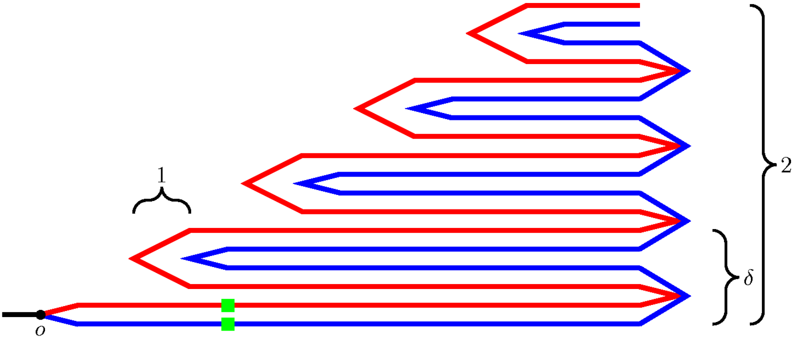

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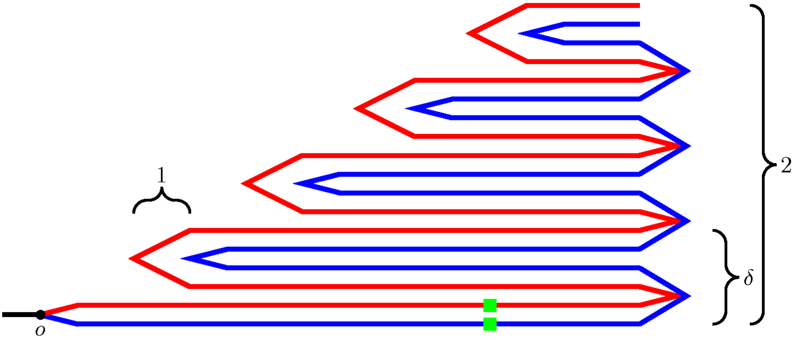

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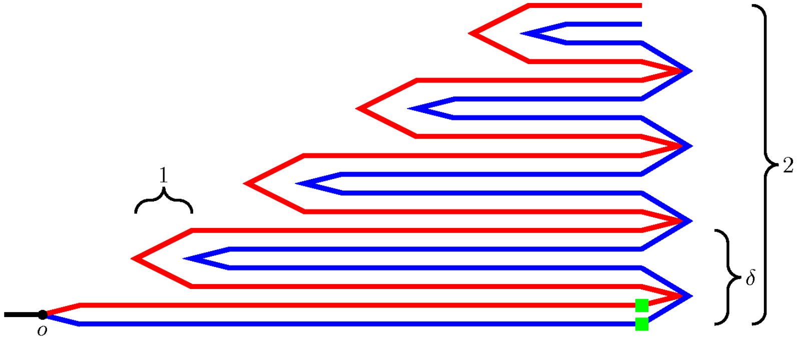

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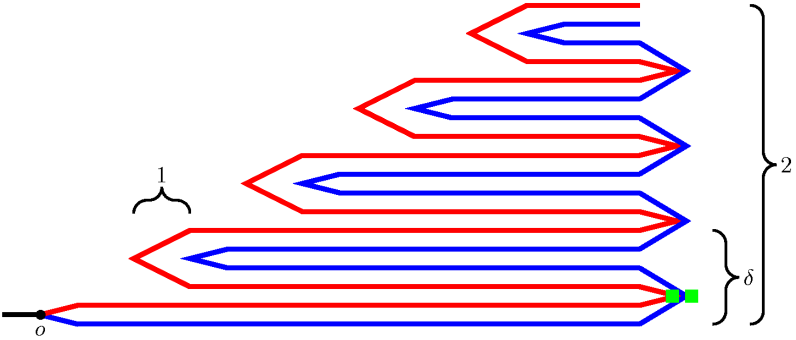

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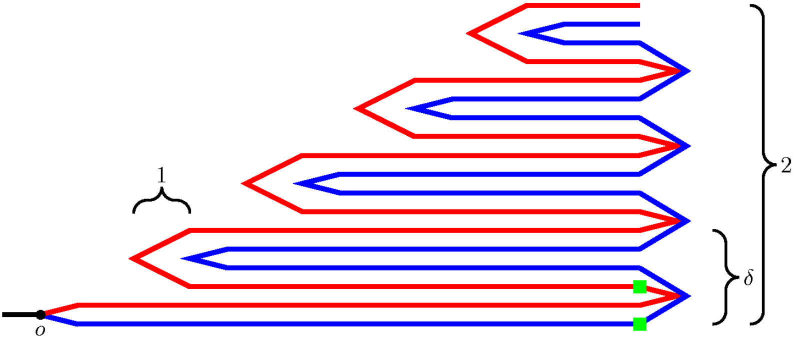

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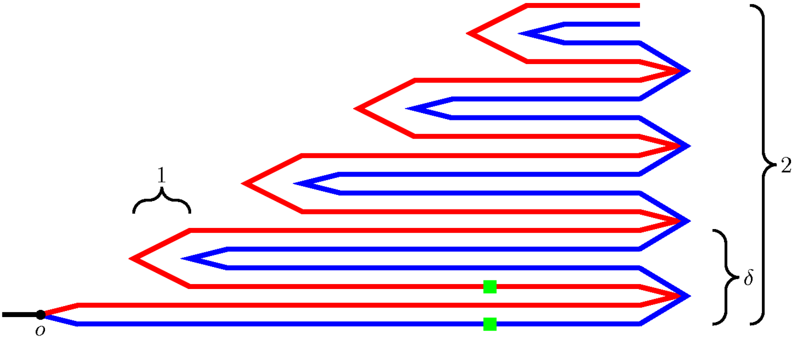

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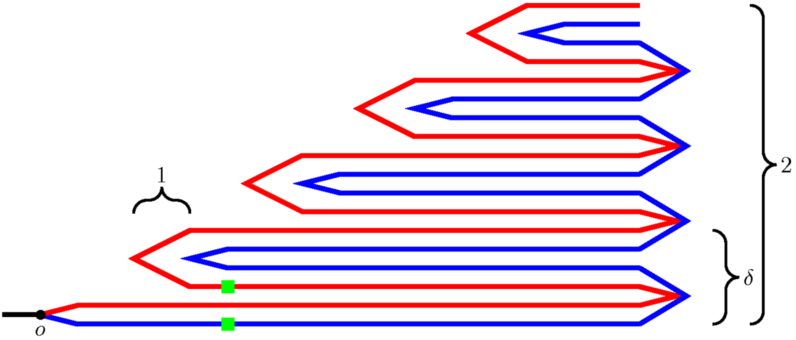

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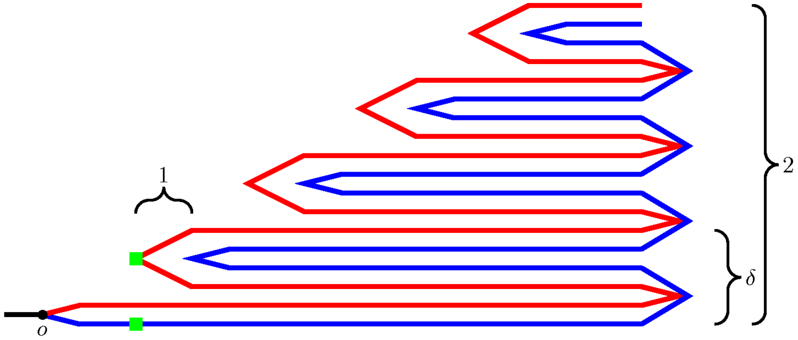

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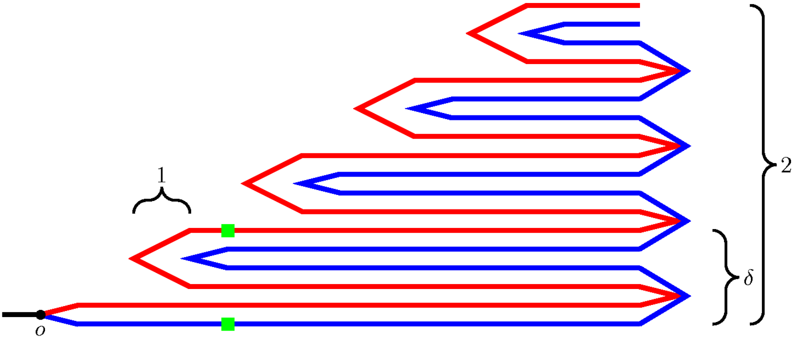

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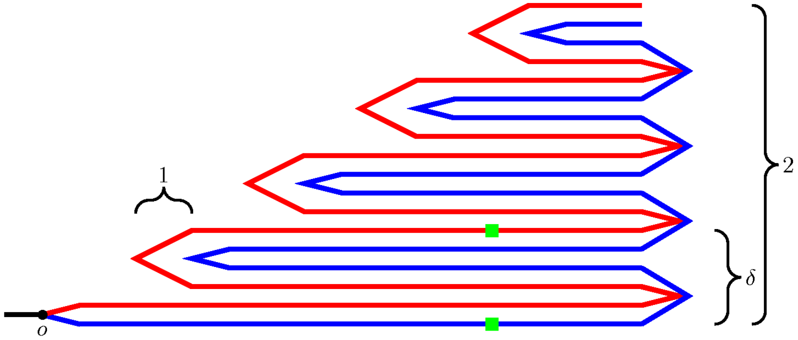

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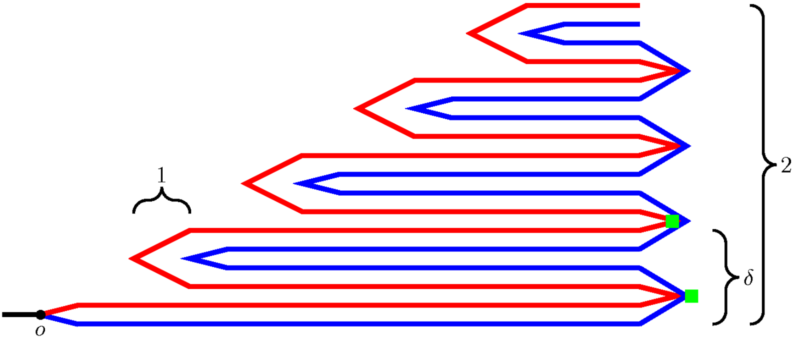

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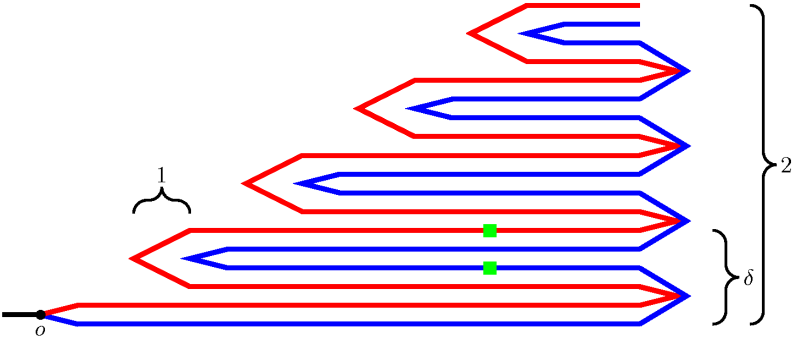

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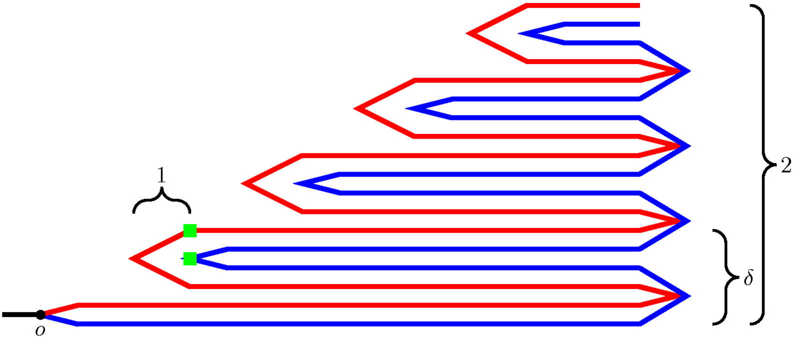

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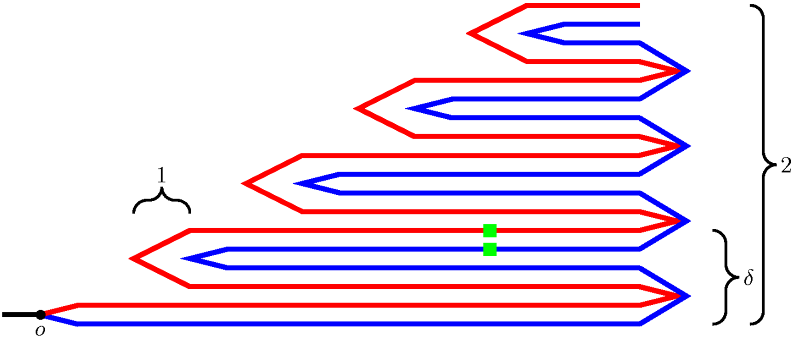

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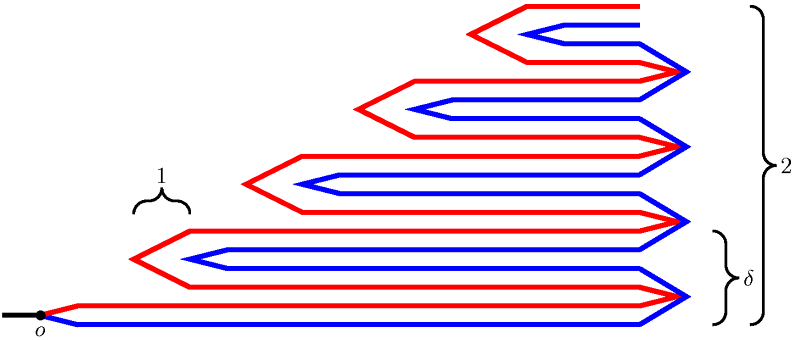

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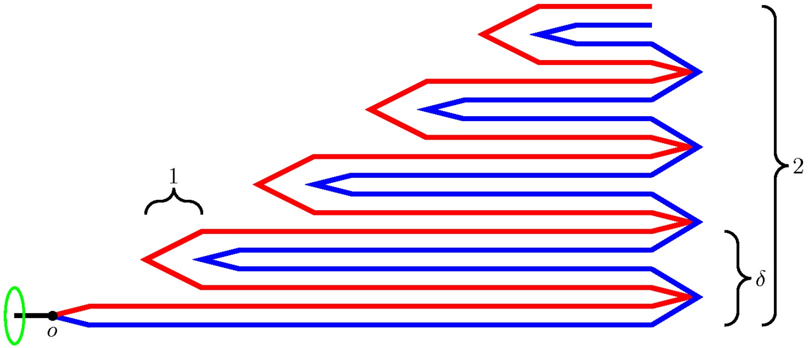

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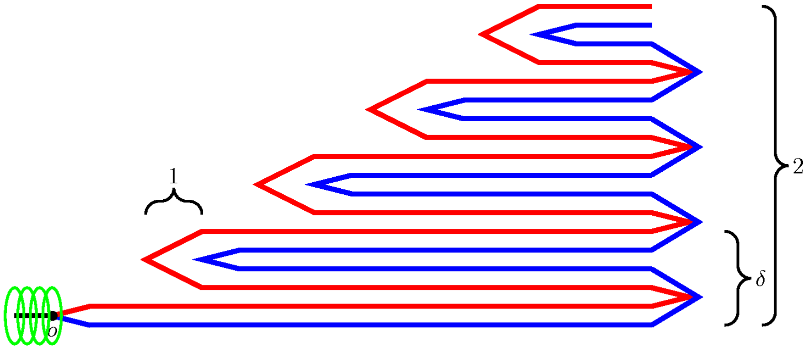

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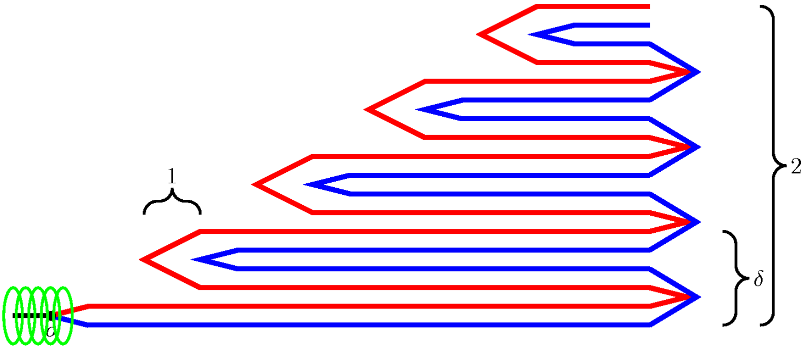

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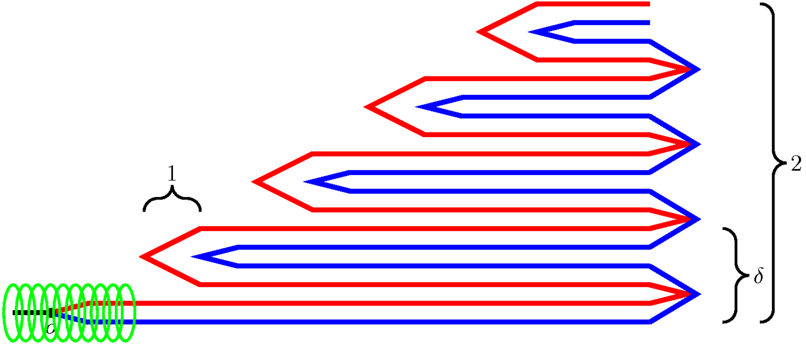

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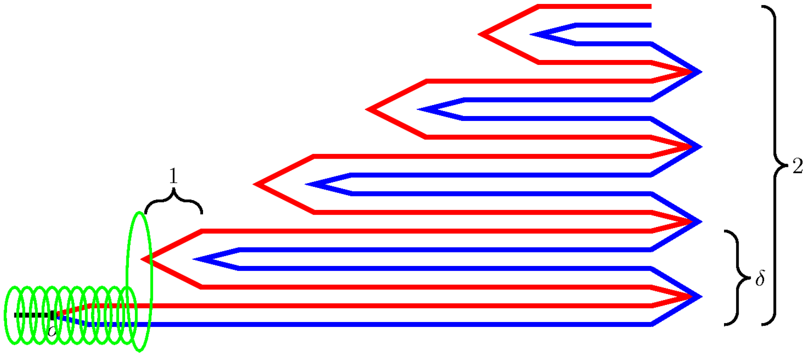

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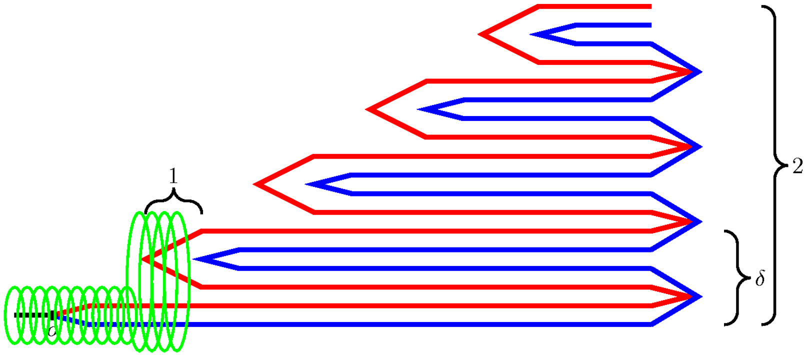

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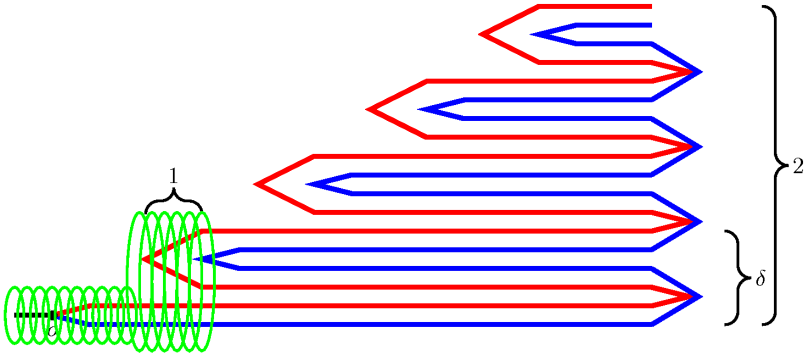

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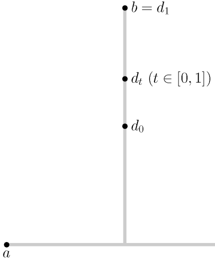
Constructing the continuum



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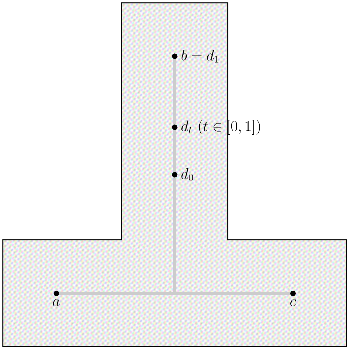

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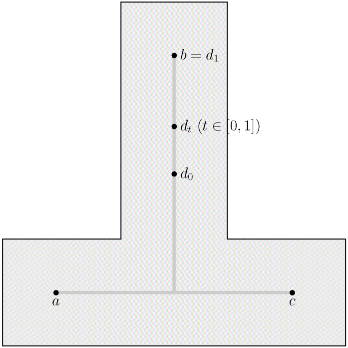

Example: consider the pattern $\begin{cases} abc \\ ad_0c \\ ac \end{cases}$

(Ingram, 1972)

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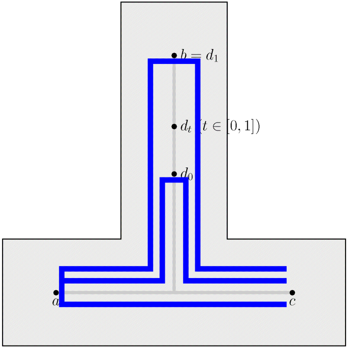

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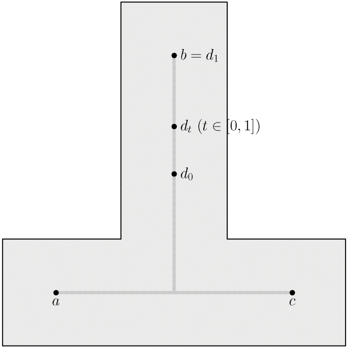

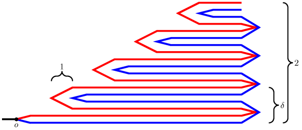

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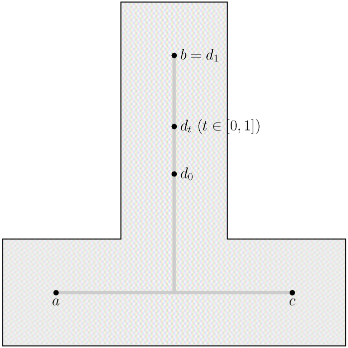

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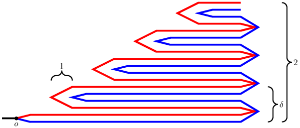

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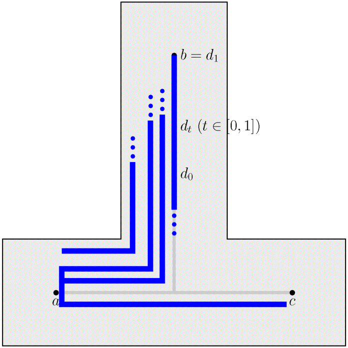
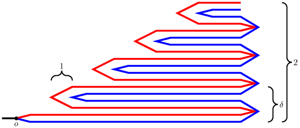
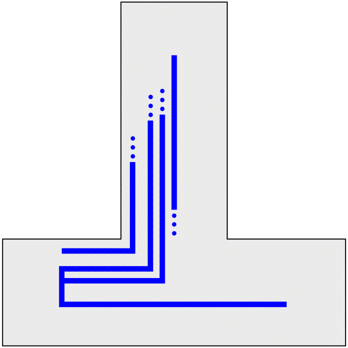


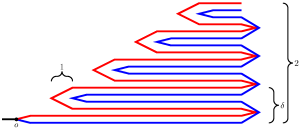
$$\begin{cases} (abc)^{2N+1}[\prod_{i=0}^{2N-1} ad_{i/2N}cd_{i/2N}a(cba)^{2N-i-1}cbc(abc)^{2N-i-1}]ad_1cd_1a(cba)^{2N+1} \\ (abc)^{2N+1}[\prod_{i=0}^{2N-1} ad_{i/2N}cd_{i/2N}a(cba)^{2N-i-1}cbabc(abc)^{2N-i-1}]ad_1cd_1a(cba)^{2N+1}cb \\ ac \end{cases}$$



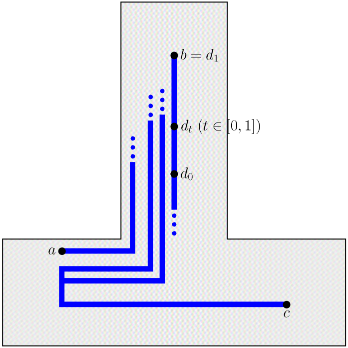
Constructing the continuum



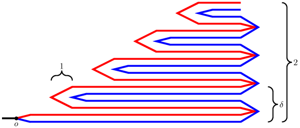
$$\begin{cases} (abc)^{2N+1}[\prod_{i=0}^{2N-1} ad_{i/2N}cd_{i/2N}a(cba)^{2N-i-1}cbc(abc)^{2N-i-1}]ad_1cd_1a(cba)^{2N+1} \\ (abc)^{2N+1}[\prod_{i=0}^{2N-1} ad_{i/2N}cd_{i/2N}a(cba)^{2N-i-1}cbabc(abc)^{2N-i-1}]ad_1cd_1a(cba)^{2N+1}cb \\ ac \end{cases}$$



Constructing the continuum



$$\begin{cases} (abc)^{2N+1}[\prod_{i=0}^{2N-1} ad_{i/2N}cd_{i/2N}a(cba)^{2N-i-1}cbc(abc)^{2N-i-1}]ad_1cd_1a(cba)^{2N+1} \\ (abc)^{2N+1}[\prod_{i=0}^{2N-1} ad_{i/2N}cd_{i/2N}a(cba)^{2N-i-1}cbabc(abc)^{2N-i-1}]ad_1cd_1a(cba)^{2N+1}cb \\ ac \end{cases}$$



Other examples?

Question

Is there a *hereditarily indecomposable* non-chainable continuum with span zero?

Other examples?

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Is there a *hereditarily indecomposable* non-chainable continuum with span zero?

Thank you!