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A non-chainable plane continuum with span zero

by

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Abstract. A plane continuum is constructed which has span zero but is not chainable.

1. Introduction

1.1. Background. The notion of the span of a continuum was introduced by Lelek in [8]. There he proved that chainable continua have span zero, and in 1971 ([9]) he asked whether the converse also holds. This is known as *Lelek's problem*, and has become a topic of much interest in continuum theory, in part because there are few other means presently available to decide whether a given continuum is chainable. An affirmative answer to Lelek's problem would have provided a useful tool with applications to other open problems in continuum theory; for instance, it would have completed the classification of planar homogeneous continua (see [20]).

Lelek's problem has been featured in a number of recent surveys, appearing as Problem 8 in [4], Problem 2 in [7], Problem 81 in [5], Conjecture 2 in [12], and in [15, p. 255].

There has been previous work toward finding a counterexample for Lelek's problem. Repovš et al. exhibit in [21] a sequence of trees in the plane with arbitrarily small (positive) spans, none of which has a chain cover of mesh < 1. In [1], Bartošová et al. consider generalizations of the notions of chainability and span zero to the class of Hausdorff (not necessarily metrizable) continua, and prove via a model-theoretic construction that a counterexample for Lelek's problem in that context would imply that there exists a metric counterexample.

Many positive partial results for Lelek's problem have been obtained in [13], [16], [17], and [20]. Notably, Minc proves in [13] that span zero is equivalent to chainability among those continua which are inverse limits of

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trees with simplicial bonding maps, and Oversteegen does the same in [16] for continua which are the image of a chainable continuum under an induced map.

A number of properties of chainable continua have been established for span zero continua. It is known that span zero continua are atriodic [8], and Oversteegen and Tymchatyn show in [19] that they are tree-like and weakly chainable. Further, Marsh proves in [11] that products of span zero continua have the fixed point property, and Bustamante et al. prove in [3] theorems about fixed point and universality properties in the hyperspace of subcontinua of a span zero continuum, generalizing corresponding theorems for chainable continua.

In this paper, we give an example showing that in general span zero does not imply chainable, even among continua in the plane. This example also provides a negative answer to a question of Mohler (Problem 171 of [5], Problem 7 of [10], and Problem 32 of [22]), which asks whether every weakly chainable atriodic tree-like continuum is chainable.

The example presented here is a simple-triod-like continuum, which we will develop as a nested intersection of thickened simple triods in the plane. In Section 2, we introduce some terminology that is useful for describing these simple triods in a combinatorial way. We then show in Section 3 how to extract combinatorial information from a given chain cover of a graph described this way (see [16] for some related work). Section 4 contains the necessary combinatorial lemmas pertaining to our particular graphs, and in Section 5 we construct the example precisely and prove it has the stated properties.

1.2. Definitions and notation. A *continuum* is a compact connected metric space. We will always denote the metric by d.

Given a continuum X, the *span* of X is the supremum of all $\eta \geq 0$ for which there exists a subcontinuum Z of $X \times X$ such that: 1) $d(x, y) \geq \eta$ for each $(x, y) \in Z$; and 2) $\pi_1(Z) = \pi_2(Z)$, where $\pi_1, \pi_2 : X \times X \to X$ are the first and second coordinate projections, respectively.

The following properties are straightforward (see [8]):

- if X and Y are continua with $X \subseteq Y$, then $\mathrm{span}(X) \leq \mathrm{span}(Y)$;
- the arc $[0, 1]$ has span zero;
- if $\langle X_n \rangle_{n=1}^{\infty}$ is a sequence of continua in a given compact metric space, then

$$\limsup_{n\to\infty} \mathrm{span}(X_n) \leq \mathrm{span}(\limsup_{n\to\infty} X_n).$$

The third property implies in particular that given any continuum $X \subset \mathbb{R}^2$ and any $\varepsilon > 0$, there is some $\delta > 0$ such that $\mathrm{span}(\overline{X_\delta}) < \mathrm{span}(X) + \varepsilon$, where X_δ denotes the δ-neighborhood of X in $\mathbb{R}^2$.

A *chain cover* of a continuum X is a finite open cover $\mathcal{U} = \langle U_\ell : 0 \leq \ell < L \rangle$ which is enumerated in such a way that $U_{\ell_1} \cap U_{\ell_2} \neq \emptyset$ if and only if $|\ell_1 - \ell_2| \leq 1$. We call X *chainable* if every open cover of X has a refinement which is a chain cover.

A *simple triod* is a continuum T which is the union of three arcs, A_1, A_2, A_3, which have a common endpoint o and are otherwise pairwise disjoint. A_1, A_2, A_3 are called the *legs* of T, and o is the *branch point* of T.

If $f : X \to Y$ is a function and $x_1, \ldots, x_n \in X$, we will often write

$$x_1 \cdots x_n \overset{f}{\mapsto} y_1 \cdots y_n$$

to mean $f(x_i) = y_i$ for each i.

Given a set S, a *total quasi-order* on S is a binary relation $\leq$ on S which is reflexive and transitive, and such that for every $s_1, s_2 \in S$, we have $s_1 \leq s_2$ or $s_2 \leq s_1$ (or both). If $\leq$ is a total quasi-order, we write $s_1 \simeq s_2$ to mean $s_1 \leq s_2$ and $s_2 \leq s_1$, and we write $s_1 < s_2$ to mean $s_1 \leq s_2$ and $s_2 \not\leq s_1$. Elements s_1 and s_2 are $\leq$-*adjacent* if $s_1 \not\simeq s_2$ and there is no $s \in S$ with $s_1 < s < s_2$ or $s_2 < s < s_1$.

If S is finite and $\leq$ is a total quasi-order on S, then there is a function $f : S \to \mathbb{Z}$ which is order preserving (i.e. $f(s_1) \leq f(s_2)$ if and only if $s_1 \leq s_2$) whose range is a contiguous block of integers.

By a *graph*, we will mean an undirected connected graph without multiple edges joining the same pair of vertices, and without any edge from a vertex to itself. If G is a graph, $V(G)$ denotes the set of vertices. A pair of vertices $v_1, v_2 \in V(G)$ is *adjacent in* G provided there is an edge between them. A sequence of distinct vertices $v_1, \ldots, v_n \in V(G)$ is *consecutive in* G provided there is an edge between v_i and v_{i+1} for each $0 \leq i \leq n-1$.

A graph G will be considered as a topological space in the usual way, where the edges are realized by arcs. If $v_1, v_2 \in V(G)$ are adjacent in G, then we will use the notation $[v_1, v_2]_G$ for the arc joining v_1 and v_2; we will often drop the subscript G when the graph is clear from the context.

A *tree* is a continuum-theoretic graph (i.e. a connected finite union of arcs with pairwise finite intersections) which contains no circle. If T is a tree and $a, b \in T$, then $[a, b]_T$ denotes the minimal arc $A \subseteq T$ with $a, b \in A$; again, we will often drop the subscript T when the tree is clear from the context.

By a *word*, we will mean a finite sequence of symbols from some alphabet Γ, where an *alphabet* is any set of symbols not including (,), [,], $\prod$, $\pitchfork$, and $^{\leftarrow}$. If ω is a word, then $|\omega|$ denotes the length of ω. A word ω will be considered as a function on the set of integers $\{0, 1, \ldots, |\omega| - 1\}$. We write $\omega^{\leftarrow}$ for the reverse of ω, defined by $\omega^{\leftarrow}(j) = \omega(|\omega| - j - 1)$.

If $\omega_1, \ldots, \omega_n$ is a sequence of words, then $\prod_{i=1}^{n} \omega_i$ denotes the concatenation of these words. If ω is a word and n is a non-negative integer, then

ω^n denotes the word obtained by repeating ω n times; that is, $\omega^n = \prod_{i=1}^{n} \omega$. Given words ω_1, ω_2 such that the last symbol of ω_1 coincides with the first symbol of ω_2, define $\omega_1 \pitchfork \omega_2$ to be the word obtained by concatenating onto ω_1 all but the first symbol in ω_2. For example, $abc \pitchfork caba = abcaba$.

2. Graph-words

2.1. Sketches and the graph-word ρ_N

Definition. A *graph-word in the alphabet Γ* is a pair $\rho = \langle G_\rho, w_\rho \rangle$ where G_ρ is a graph, and $w_\rho : V(G_\rho) \to \Gamma$ is a function.

Let us fix, for the rest of this paper, the alphabet $\Gamma := \{a, b, c\} \cup \{b_t : t \in [0,1]\}$.

For each positive integer N, denote by α_N, β_N, γ_N the following three words:

$$(abc)^{2N+1} \prod_{i=0}^{2N-1} [ab_{i/2N}cb_{i/2N}a(cba)^{2N-i-1}cbc(abc)^{2N-i-1}]\ ab_1cb_1a(cba)^{2N+1},$$

$$(abc)^{2N+1} \prod_{i=0}^{2N-1} [ab_{i/2N}cb_{i/2N}a(cba)^{2N-i-1}cbabc(abc)^{2N-i-1}]\ ab_1cb_1a(cba)^{2N+1}cb,$$

$$ac.$$

For later use, we also define the word β_N^- to be identical to β_N except without the final b.

Define the graph-word ρ_N as follows. Let G_{ρ_N} be a simple triod, with vertex set $V(G_{\rho_N}) = \{o, p_1, \ldots, p_{|\alpha_N|-1}, q_1, \ldots, q_{|\beta_N|-1}, r\}$, where o is the branch point of the triod, $p_{|\alpha_N|-1}, q_{|\beta_N|-1}, r$ are the endpoints of G_{ρ_N}, the points p_j belong to the leg $[o, p_{|\alpha_N|-1}]$ with $p_j \in [o, p_{j+1}]$ for each j, and the points q_j belong to the leg $[o, q_{|\beta_N|-1}]$ with $q_j \in [o, q_{j+1}]$ for each j. Put $p_0 := o$ and $q_0 := o$. Define $w_{\rho_N} : V(G_{\rho_N}) \to \Gamma$ by $w_{\rho_N}(p_j) := \alpha_N(j)$, $w_{\rho_N}(q_j) := \beta_N(j)$, and $w_{\rho_N}(r) := \gamma_N(1) = c$.

To construct the example of a non-chainable continuum X with span zero, we will define a sequence of simple triods $\langle T_N \rangle_{N=0}^{\infty}$ in the plane such that T_N is contained in a small neighborhood of T_{N-1} in $\mathbb{R}^2$ for each $N > 0$; X will then be defined as the intersection of the nested sequence of neighborhoods of the triods T_N. The graph-word ρ_N will be used to describe the pattern with which we nest the simple triod T_N inside a small neighborhood of T_{N-1}. To carry this out precisely, we introduce the notion of a *sketch* below.

Remark. The space X may alternatively be described as an inverse limit of simple triods, as follows. Let T be a simple triod with endpoints denoted as a, b, c and branch point o. Denote a point in the interior of the arc $[o, b]$ by b_0, and parameterize the arc $[b_0, b]$ by b_t for $t \in [0,1]$, so that

$b_1 = b$ (as per the notion of a Γ-marking defined below). Then the Nth bonding map $b_N : T \to T$ takes o to a, is the identity on the segment $[b_0, b]$, and otherwise maps the legs $[o,a]$, $[o,b]$, $[o,c]$ in a piecewise linear way according to the patterns α_N, β_N, γ_N, respectively. Figures 1–3, along with the proof of Proposition 1 below, provide some geometric intuition for how this looks.

Definition. Given a simple triod T with branch point o, a *Γ-marking of T* is a function $\iota : \Gamma \to T$ such that $\iota(a)$, $\iota(b)$, $\iota(c)$ are the endpoints of T and $\{\iota(b_t) : t \in [0,1]\} \subset [o, \iota(b)]$ are such that whenever $t < t'$, we have $\iota(b_t) \in [o, \iota(b_{t'})]$ and $\operatorname{diam}([\iota(b_t), \iota(b_{t'})]) = d(\iota(b_t), \iota(b_{t'})) = t' - t$.

Define the simple triod $T_0 := \{(x,0) : x \in [-1,1]\} \cup \{(0,y) : y \in [0,2]\} \subset \mathbb{R}^2$, and define a Γ-marking $\iota : \Gamma \to T_0$ by

$$\iota(a) := (-1,0), \qquad \iota(b) := (0,2), \qquad \iota(c) := (1,0),$$
$$\iota(b_t) := (0, 1+t) \text{ for } t \in [0,1].$$

Definition. Define the equivalence relation $\approx_\Gamma$ on Γ by $\sigma \approx_\Gamma \tau$ if and only if $\sigma = \tau$ or $\sigma, \tau \in \{b\} \cup \{b_t : t \in [0,1]\}$.

The relation $\approx_\Gamma$ partitions Γ into three equivalence classes. If ι is a Γ-marking of a triod T, then $\sigma \approx_\Gamma \tau$ if and only if $\iota(\sigma)$ and $\iota(\tau)$ belong to the same leg of T.

To simplify definitions and arguments in the following, we will restrict our attention to a special class of graph-words.

Definition. A *compliant graph-word* is a graph-word $\langle G, w\rangle$ in the alphabet Γ such that there is no pair of adjacent vertices v_1, v_2 in G with $w(v_1) \approx_\Gamma w(v_2)$.

Observe that ρ_N is a compliant graph-word for each N.

Definition. Suppose T is a simple triod with a Γ-marking $\iota : \Gamma \to T$, and let $\rho = \langle G, w\rangle$ be a compliant graph-word in the alphabet Γ. Then $\widehat{w} : G \to T$ is a *ρ-suggested bonding map* provided $\widehat{w}|_{V(G)} = \iota \circ w$, and for any adjacent $v_1, v_2 \in V(G)$, the restriction $\widehat{w}|_{[v_1,v_2]_G}$ is a homeomorphism from $[v_1, v_2]_G$ to $[\iota(w(v_1)), \iota(w(v_2))]_T$.

Definition. Let $\langle \Omega, d\rangle$ be a metric space, let $T \subseteq \Omega$ be a Γ-marked simple triod, let $G \subseteq \Omega$ be a graph, and let $\varepsilon > 0$. Then $\rho = \langle G, w\rangle$ is a *$\langle T, \varepsilon\rangle$-sketch of G in Ω* if ρ is a compliant graph-word in the alphabet Γ, and there is a ρ-suggested bonding map $\widehat{w} : G \to T$ such that $d(x, \widehat{w}(x)) < \varepsilon/2$ for every $x \in G$.

The next proposition assures us that we may use the graph-word ρ_N defined above to describe the pattern with which we embed one simple triod into a small neighborhood of another, in the plane.

We will need some additional notation when working with the graph-word ρ_N. For each $i \leq 2N$, define $n(i)$ and $m(i)$ to be the unique integers such that

$$(n(i)-1)\ n(i)\ (n(i)+1) \xmapsto{\alpha_N} b_{i/2N} c b_{i/2N},$$
$$(m(i)-1)\ m(i)\ (m(i)+1) \xmapsto{\beta_N} b_{i/2N} c b_{i/2N}.$$

For each $i < 2N$, define $\theta(i) := 6N - 3i + 1$ and $\phi(i) := 6N - 3i + 2$. Observe that

$$(n(i)+\theta(i)-1)\ (n(i)+\theta(i))\ (n(i)+\theta(i)+1) \xmapsto{\alpha_N} cbc,$$
$$(m(i)+\phi(i)-2)\ (m(i)+\phi(i)-1)\cdots(m(i)+\phi(i)+2) \xmapsto{\beta_N} cbabc.$$

Note that $n(0) = m(0) = 6N + 5$, and that $n(i) + 2\theta(i) = n(i+1)$ and $m(i) + 2\phi(i) = m(i+1)$ for each $i < 2N$.

When discussing an embedding of the graph G_{ρ_N} in $\mathbb{R}^2$, we will make no notational distinction between the points of G_{ρ_N} before and after the embedding.

Proposition 1. *Suppose $T \subset \mathbb{R}^2$ is a simple triod and $\iota : \Gamma \to T$ is a Γ-marking. For any integer $N > 0$ and any $\varepsilon > 0$, there is an embedding of the simple triod graph G_{ρ_N} in $\mathbb{R}^2$ such that ρ_N is a $\langle T, \varepsilon\rangle$-sketch of G_{ρ_N} in $\mathbb{R}^2$. Moreover, the embedding can be chosen such that $q_{|\beta_N|-1} = \iota(b)$, $q_{|\beta_N|-2} = \iota(c)$, and $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{G_{\rho_N}} = [\iota(c), \iota(b)]_T$.*

Proof. For simplicity, we will argue only the case $T = T_0$, with the Γ-marking ι as described above; the general case can be treated similarly.

First we will analytically define a different embedding of G_{ρ_N} in $\mathbb{R}^2$, then we will describe how to obtain the desired embedding from it.

Let $\eta > 0$ be significantly smaller than ε, say $\eta < \varepsilon/20N^2$. For $0 \leq i \leq 2N$, put

$$p_{n(i)} := (1+\eta, (4i+3/2)\eta), \qquad q_{m(i)} := (1, (4i+3/2)\eta).$$

For $0 \leq i < 2N$ and $1 \leq j < \theta(i)$, put

$$p_{n(i)+j} := (1-j, (4i+3)\eta), \qquad p_{n(i+1)-j} := (1-j, 4(i+1)\eta),$$

and put $p_{n(i)+\theta(i)} := (1-\theta(i), (4i+7/2)\eta)$. For $0 \leq i < 2N$ and $1 \leq j < \phi(i)$, put

$$q_{m(i)+j} := (1-j, (4i+2)\eta), \quad q_{m(i+1)-j} := (1-j, (4(i+1)+1)\eta),$$

and put $q_{m(i)+\phi(i)} := (1-\phi(i), (4i+7/2)\eta)$. Further, put

$$p_{n(0)-j} := (1-j, 0) \qquad \text{for } 1 \leq j < 6N+5,$$
$$q_{m(0)-j} := (1-j, \eta) \qquad \text{for } 1 \leq j < 6N+5,$$

$$q_{m(2N)+j} := (1-j, (8N+2)\eta) \quad \text{for } 1 \le j \le 6N+7,$$
$$p_{n(2N)+j} := (1-j, (8N+3)\eta) \quad \text{for } 1 \le j \le 6N+5.$$

Finally, put $o := \left(-6N-4, \frac{1}{2}\eta\right)$ and $r := \left(-6N-5, \frac{1}{2}\eta\right)$. Join each pair of adjacent vertices in G_{ρ_N} by a straight line segment in $\mathbb{R}^2$. Denote the resultant embedding of G_{ρ_N} in $\mathbb{R}^2$ by G'. Figure 1 depicts the embedding G' for $N = 1$.

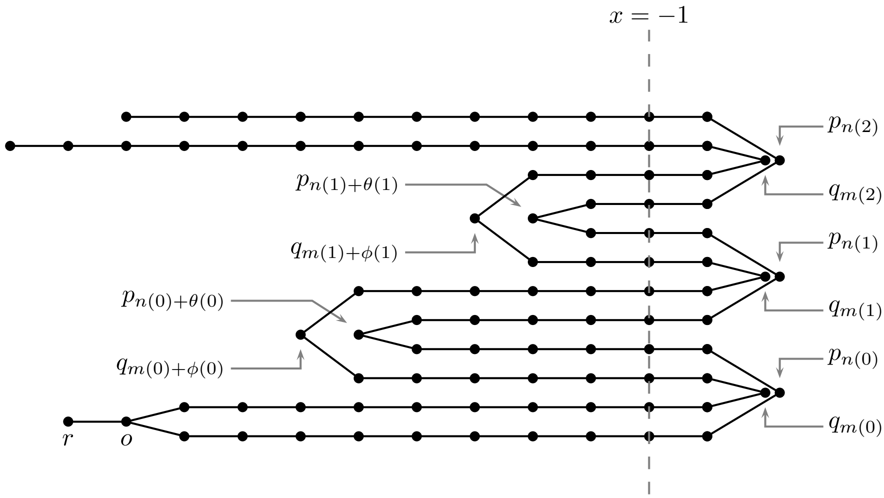


Fig. 1. The intermediate stage G' for the embedding of G_{ρ_1} in $\mathbb{R}^2$

Observe that in G', for each integer $k \le -1$, if v and v' are two vertices in the line $x = k$, then $w(v) = w(v')$. Also notice that each vertex v in the line $x = -1$ is already close to the point $\iota(w(v)) = \iota(a) = (-1, 0)$, and that each vertex u of the form $p_{n(i)}$ or $q_{m(i)}$ is already close to the point $\iota(w(u)) = \iota(c) = (1, 0)$. We now describe heuristically in two steps how to mold G' into the embedding we seek.

First, for each $i \le 2N$, for each triple $\langle v_1, v_2, v_3\rangle$ of the form $\langle p_{n(i)-2}, p_{n(i)-1}, p_{n(i)}\rangle$, $\langle q_{m(i)-2}, q_{m(i)-1}, q_{m(i)}\rangle$, $\langle p_{n(i)+2}, p_{n(i)+1}, p_{n(i)}\rangle$, or $\langle q_{m(i)+2}, q_{m(i)+1}, q_{m(i)}\rangle$, move the vertex v_2 up to be close to the point $\iota(b_{i/2N})$, move the vertex v_3 down slightly, and shape the arcs joining v_1 to v_2 and v_2 to v_3 so that:

(1) there is a homeomorphism $\hat{w}_1 : [v_1, v_2]_{G'} \to [\iota(a), \iota(b_{i/2N})]_{T_0}$ such that $\hat{w}_1(v_1) = \iota(a)$, $\hat{w}_1(v_2) = \iota(b_{i/2N})$, and $d(x, \hat{w}_1(x)) < \eta$ for each $x \in [v_1, v_2]_{G'}$,

(2) there is a homeomorphism $\hat{w}_2 : [v_2, v_3]_{G'} \to [\iota(b_{i/2N}), \iota(c)]_{T_0}$ such that $\hat{w}_2(v_2) = \iota(b_{i/2N})$, $\hat{w}_2(v_3) = \iota(c)$, and $d(x, \hat{w}_2(x)) < \eta$ for each $x \in [v_2, v_3]_{G'}$,

(3) $[v_1, v_2]_{G'} \cup [v_2, v_3]_{G'}$ misses the closed upper-right quadrant of the plane, $\{(x, y) : x \geq 0,\ y \geq 0\}$,

and so that in the end no new intersections between those arcs have been introduced (i.e., so that the result is still an embedding of G_{ρ_N}). Figure 2 depicts the result for $N = 1$.

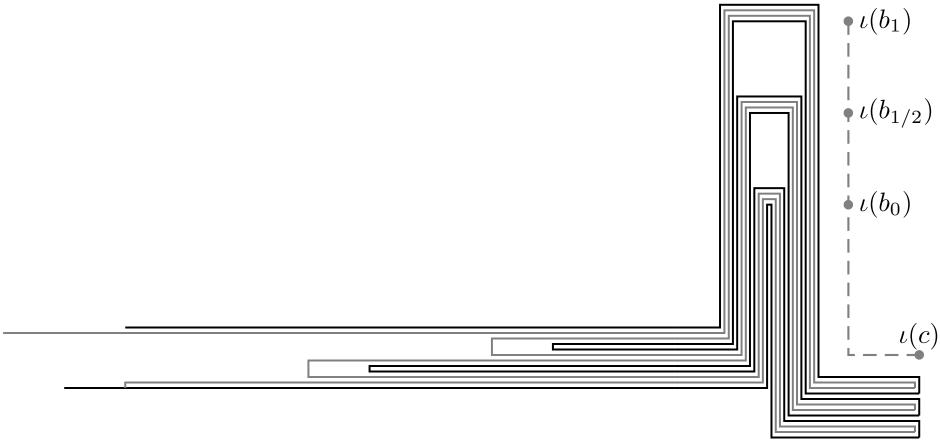
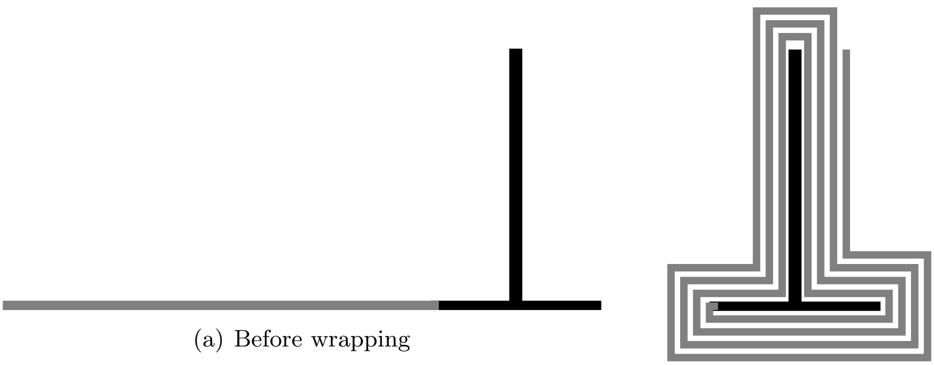


Fig. 2. The second intermediate stage for the embedding of G_{ρ_1} in $\mathbb{R}^2$

Next, take the strip $\{(x, y) : x \leq -1,\ 0 \leq y \leq (8N + 3)\eta\}$ and stretch and wind it counter-clockwise $2N + 2$ times around the outside of

$$\bigcup_{i=0}^{2N} ([p_{n(i)-2}, p_{n(i)+2}]_{G'} \cup [q_{m(i)-2}, q_{m(i)+2}]_{G'}),$$



(a) Before wrapping

(b) After wrapping

Fig. 3. Wrapping the strip counterclockwise around the simple triod to obtain the embedding of G_{ρ_N} in $\mathbb{R}^2$

so that for each integer $k \leq -1$, all the vertices v in the line $x = k$ end up near the point $\iota(w(v)) \in T_0$, taking care to make sure $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{G'} = [\iota(c), \iota(b)]_{T_0}$. Figure 3 depicts roughly how this wrapping looks.

The resulting embedding has the desired properties. ∎

2.2. Span and ρ_N. In this section we prove that the span of a simple triod embedded in a way described by ρ_N converges to 0 as $N \to \infty$. This will ensure that we will obtain a continuum with span zero when we take the nested intersection of neighborhoods of triods embedded in $\mathbb{R}^2$ as described by the ρ_N's.

Given a set T and a subset $S \subseteq T \times T$, let $S^{-1} = \{(x, y) \in T \times T : (y, x) \in S\}$.

Lemma 2. *Let T be a simple triod with legs A_1, A_2, A_3 and branch point o. For each $i \in \{1, 2, 3\}$ let p_i be the endpoint of leg A_i other than o. Suppose $\delta > 0$ and $W \subset A_1 \times A_2$ is an arc such that $(o, o) \in W$, W meets $(\{p_1\} \times A_2) \cup (A_1 \times \{p_2\})$, and $d(x_1, x_2) \leq \delta$ for each $(x_1, x_2) \in W$. Then the span of T is $\leq \delta$.*

Proof. Suppose $Z \subset T \times T$ is a subcontinuum with $\pi_1(Z) = \pi_2(Z)$. If $\pi_1(Z)$ is an arc, then it is easy to see that Z meets the diagonal $\Delta T = \{(x, x) : x \in T\}$, as arcs have span zero.

If $\pi_1(Z)$ is a subtriod T' of T, then we may assume $T = T'$ by replacing the arc W by the component of $W \cap (T' \times T')$ that contains (o, o). Let K_1 and K_2 be disjoint closed and open subsets of $(A_1 \times A_2) \setminus W$ such that $(A_1 \times \{o\}) \setminus W \subset K_1$, $(\{o\} \times A_2) \setminus W \subset K_2$, and $K_1 \cup K_2 = (A_1 \times A_2) \setminus W$.

For each $i \in \{1, 2, 3\}$ let U_i and V_i be the two components of $(A_i \times A_i) \setminus \Delta T$, where $(A_i \setminus \{o\}) \times \{o\} \subset U_i$ and $\{o\} \times (A_i \setminus \{o\}) \subset V_i$. It can then be seen that the set

$$Y := (U_1 \cup U_2 \cup V_3 \cup (A_1 \times A_3) \cup (A_2 \times A_3) \cup K_1 \cup K_2^{-1}) \setminus W$$

is closed and open in $(T \times T) \setminus (W \cup W^{-1} \cup \Delta T)$ (see Proposition 5.1 of [6]). Note that $(T \times T) \setminus (W \cup W^{-1} \cup \Delta T) = Y \cup Y^{-1}$.

Observe that $p_3 \notin \pi_1(Y)$ and $p_3 \notin \pi_2(Y^{-1})$, hence $Z \not\subseteq Y$ and $Z \not\subseteq Y^{-1}$. Since Z is connected, it follows that Z must meet $W \cup W^{-1} \cup \Delta T$.

Thus in either case, there is some $(x_1, x_2) \in Z$ with $d(x_1, x_2) \leq \delta$. Therefore T has span $\leq \delta$. ∎

Proposition 3. *Suppose $T \subset \mathbb{R}^2$ is Γ-marked. If the triod graph G_{ρ_N} is embedded in $\mathbb{R}^2$ such that ρ_N is a $\langle T, \varepsilon \rangle$-sketch of G_{ρ_N} in $\mathbb{R}^2$, then the span of G_{ρ_N} is less than $1/2N + \varepsilon$.*

Proof. To apply Lemma 2, we will produce an arc $W \subset [o, p_{|\alpha_N|-1}] \times [o, q_{|\beta_N|-1}]$. Intuitively, one may think of W as a pair of points traveling simultaneously, one on the leg $[o, p_{|\alpha_N|-1}]$ and the other on $[o, q_{|\beta_N|-1}]$, starting with both at o, ending with one at the end of its leg, and at every moment staying within distance $1/2N+\varepsilon$ from one another. With this in mind, and referring to Figure 1, one should be easily convinced that such a W may be defined which passes through the following pairs, in order: (o,o), $(p_{n(0)}, q_{m(0)})$, $(p_{n(0)-\phi(0)}, q_{m(0)+\phi(0)})$, $(p_{n(0)}, q_{m(1)})$, $(p_{n(0)+\theta(0)}, q_{m(1)-\theta(0)})$, $(p_{n(1)}, q_{m(1)})$, $\ldots$, $(p_{n(2N)}, q_{m(2N)})$, $(p_{|\alpha_N|-1}, q_{m(2N)+6N+5})$. The precise definition of this arc W follows.

Suppose that n,n' and m,m' are two pairs of adjacent integers. Let $S^{n,n'}_{m,m'}$ denote the square $[p_n,p_{n'}]\times[q_m,q_{m'}]$. Suppose one of the following occurs:

(1) $w(p_n)=w(q_m)$, $w(p_{n'})=w(q_{m'})$;
(2) $w(p_n)=w(q_m)$, $w(p_{n'})=b_{i/2N}$, $w(q_{m'})=b_{(i+1)/2N}$ for some i; or
(3) $w(p_{n'})=w(q_{m'})$, $w(p_n)=b_{i/2N}$, $w(q_m)=b_{(i+1)/2N}$ for some i.

Then let $W^{n,n'}_{m,m'}\subset S^{n,n'}_{m,m'}$ be an arc such that $d(x_1,x_2)<1/2N+\varepsilon$ for each $(x_1,x_2)\in W^{n,n'}_{m,m'}$, and $W^{n,n'}_{m,m'}\cap\partial S^{n,n'}_{m,m'}=\{(p_n,q_m),(p_{n'},q_{m'})\}$.

Define the arc $W\subset[o,p_{|\alpha_N|-1}]\times[o,q_{|\beta_N|-1}]$ as follows (it will be helpful to refer to Figure 1 when reading this formula):

$$\begin{aligned}W := \bigcup_{j=0}^{n(0)-1} W^{j,j+1}_{j,j+1} \cup \bigcup_{i=0}^{2N-1}\Bigg(&\bigcup_{j=0}^{\phi(i)-1} W^{n(i)-j,\,n(i)-j-1}_{m(i)+j,\,m(i)+j+1}\\ &\cup \bigcup_{j=0}^{\phi(i)-1} W^{n(i)-\phi(i)+j,\,n(i)-\phi(i)+j+1}_{m(i)+\phi(i)+j,\,m(i)+\phi(i)+j+1} \cup \bigcup_{j=0}^{\theta(i)-1} W^{n(i)+j,\,n(i)+j+1}_{m(i+1)-j,\,m(i+1)-j-1}\\ &\cup \bigcup_{j=0}^{\theta(i)-1} W^{n(i)+\theta(i)+j,\,n(i)+\theta(i)+j+1}_{m(i+1)-\theta(i)+j,\,m(i+1)-\theta(i)+j+1}\Bigg) \cup \bigcup_{j=0}^{6N+4} W^{n(2N)+j,\,n(2N)+j+1}_{m(2N)+j,\,m(2N)+j+1}.\end{aligned}$$

Then W contains (o,o) and meets $\{p_{|\alpha_N|-1}\}\times[o,q_{|\beta_N|-1}]$, and $d(x_1,x_2)<1/2N+\varepsilon$ for each $(x_1,x_2)\in W$, hence the claim follows by Lemma 2. ∎

3. Combinatorics from chain covers

3.1. Chain quasi-orders. The following definition is closely related to the notion of a chain word reduction from [14]. It should be thought of as follows: if $\langle G,w\rangle$ is a $\langle T,\varepsilon\rangle$-sketch of G and we have a chain cover of G of small mesh, then $v_1\le v_2$ means roughly that the chain "covers v_1 before, or at around the same time as, v_2" (see Proposition 5).

Definition. Suppose $\langle G,w\rangle$ is a compliant graph-word. A *chain quasi-order* of $\langle G,w\rangle$ is a total quasi-order $\le$ on $V(G)$ satisfying:

(C1) if $v_1\simeq v_2$, then $w(v_1)\approx_\Gamma w(v_2)$;
(C2) if $v_1,v_2\in V(G)$ are adjacent in G, then v_1 and v_2 are $\le$-adjacent;

(C3) if $v_1, v_2, v_3 \in V(G)$ are consecutive in G, $v \in V(G)$, and if $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0,1]$ are such that $t' \geq t$, $v_1v_2v_3 \overset{w}{\mapsto} \sigma b_t \tau$, $w(v) = b_{t'}$, and $v_1 < v_2 \simeq v < v_3$, then $t' - t < 1/2$.

Notice that if $\leq$ is a chain quasi-order, then the reverse order of $\leq$ (defined by $v_1 \leq^* v_2$ if and only if $v_2 \leq v_1$) is also a chain quasi-order.

The following simple lemma will be useful later on.

LEMMA 4. *Let $\leq$ be a chain quasi-order of $\langle G, w\rangle$. Suppose $v_1, s_1, \dots, s_\kappa, v_2$ are consecutive in G and $v \in V(G)$ is such that $v_1 < v < v_2$. Then there is some $i \in \{1, \dots, \kappa\}$ such that $v \simeq s_i$.*

Proof. Put $s_0 := v_1$, $s_{\kappa+1} := v_2$, and let i be the largest integer in $\{0, \dots, \kappa\}$ such that $s_i \leq v$. Then $s_{i+1} > v$, so since s_i and s_{i+1} are $\leq$-adjacent by property (C2), we must have $s_i \geq v$. Thus $s_i \simeq v$. ∎

3.2. Chain covers and the triod T_0

PROPOSITION 5. *Suppose $\langle G, w\rangle$ is a compliant graph-word which is a $\langle T_0, \varepsilon\rangle$-sketch of a graph G in $\mathbb{R}^2$, where $0 < \varepsilon < 1/2$. If there is a chain cover for G of mesh $< 1/2 - \varepsilon$, then there is a chain quasi-order of $\langle G, w\rangle$.*

Proof. Let $\mathcal{U} = \langle U_\ell : 0 \leq \ell < L\rangle$ be a chain cover for G of mesh $< 1/2-\varepsilon$, ordered so that $U_\ell \cap U_{\ell'} \neq \emptyset$ if and only if $|\ell - \ell'| \leq 1$. For each $v \in V(G)$, let $\ell(v)$ be such that $v \in U_{\ell(v)}$ (for each v there are either one or two choices for $\ell(v)$).

Observe that if $v_1, v_2 \in V(G)$ and $\ell(v_1) = \ell(v_2)$, then $w(v_1) \approx_\Gamma w(v_2)$, since otherwise $d(\iota(w(v_1)), \iota(w(v_2))) \geq \sqrt{2} > 1/2$, hence $d(v_1, v_2) > 1/2 - \varepsilon$, contradicting the condition that the diameter of $U_{\ell(v_1)} = U_{\ell(v_2)}$ is $< 1/2 - \varepsilon$.

Define the relation $\leq$ on $V(G)$ by setting $v_1 \leq v_2$ if and only if for every $v \in V(G)$ satisfying $\ell(v_2) \leq \ell(v) \leq \ell(v_1)$ we have $w(v) \approx_\Gamma w(v_1)$.

The following properties follow directly from the definition of $\leq$:

PROPERTIES.

(1) If $\ell(v_1) \leq \ell(v_2)$, then $v_1 \leq v_2$.
(2) If $v_1 \leq v_2$ and $w(v_1) \not\approx_\Gamma w(v_2)$, then $\ell(v_1) < \ell(v_2)$.
(3) If $v_1, v_2 \in V(G)$ are $\leq$-adjacent, then $w(v_1) \not\approx_\Gamma w(v_2)$.

It is straightforward to check using the definition and these properties that $\leq$ is a total quasi-order.

We now check that $\leq$ satisfies conditions (C1), (C2), and (C3) of the definition of a chain quasi-order.

(C1): Suppose $v_1, v_2 \in V(G)$ with $v_1 \simeq v_2$. Assume without loss of generality that $\ell(v_2) \leq \ell(v_1)$. It then follows immediately from the definition of $\leq$ and the assumption $v_1 \leq v_2$ that $w(v_1) \approx_\Gamma w(v_2)$ (take $v = v_2$).

(C2): Suppose $v_1, v_2 \in V(G)$ are adjacent in G. Since $\langle G, w\rangle$ is compliant, we know that $w(v_1) \not\approx_\Gamma w(v_2)$. Let $\sigma := w(v_1)$ and $\tau := w(v_2)$. Assume without loss of generality that $\ell(v_1) < \ell(v_2)$, which implies that $v_1 < v_2$.

If $v \in V(G)$ were such that $w(v) \not\approx_\Gamma \sigma, \tau$ and $v_1 < v < v_2$, then $\ell(v_1) < \ell(v) < \ell(v_2)$. This would imply that the link $U_{l(v)}$ contains the point v and meets the arc $[v_1, v_2]$. Since $\langle G, w\rangle$ is compliant, the only possible cases are:

- $\{\sigma, \tau\} = \{a, b\}$ and $w(v) = c$,
- $\{\sigma, \tau\} = \{a, c\}$ and $w(v) \in \{b\} \cup \{b_t : t \in [0, 1]\}$,
- $\{\sigma, \tau\} = \{b, c\}$ and $w(v) = a$,
- $\{\sigma, \tau\} = \{a, b_t\}$ and $w(v) = c$ (for some $t \in [0, 1]$),
- $\{\sigma, \tau\} = \{c, b_t\}$ and $w(v) = a$ (for some $t \in [0, 1]$).

In each case, we have $d(\iota(w(v)), [\iota(\sigma), \iota(\tau)]) \geq 1 > 1/2$. But this yields a contradiction, since $\mathcal{U}$ has mesh $< 1/2 - \varepsilon$.

Suppose for a contradiction that v_1, v_2 are not adjacent in the $\leq$ order. Let v, v' be such that $v_1 < v < v'$, and v_1, v are $\leq$-adjacent and v, v' are $\leq$-adjacent. By the above, we see that $w(v), w(v')$ are each $\approx_\Gamma$ to either σ or τ, hence by Property (3) the only possibility is $w(v) \approx_\Gamma \tau$, $w(v') \approx_\Gamma \sigma$. Property (2) then implies that $\ell(v_1) < \ell(v) < \ell(v')$.

Define the arc $A \subset T_0$ according to the value of σ as follows:

$$A := \begin{cases} [\iota(a), o] & \text{if } \sigma = a, \\ [\iota(c), o] & \text{if } \sigma = c, \\ [\iota(b), o] & \text{if } \sigma \in \{b\} \cup \{b_t : t \in [0, 1]\}. \end{cases}$$

In each case, observe that $d(\iota(w(v)), A) \geq 1 > 1/2$, and also $B_{1/2}(\iota(\sigma)) \subset A$ and $B_{1/2}(\iota(w(v'))) \subset A$.

Applying the homeomorphism $\widehat{w}|_{[v_1,v_2]}$ yields the chain cover $\langle \widehat{w}(U_\ell \cap [v_1, v_2]) : \ell' \leq \ell \leq \ell''\rangle$ of the arc $[\iota(\sigma), \iota(\tau)]_{T_0}$, where $\ell' := \min\{\ell : U_\ell \cap [v_1, v_2] \neq \emptyset\}$ and $\ell'' := \max\{\ell : U_\ell \cap [v_1, v_2] \neq \emptyset\}$.

Notice that $\widehat{w}(U_{\ell(v_1)})$ and $\widehat{w}(U_{\ell(v')})$ are sets of diameter $< 1/2$ containing $\iota(\sigma)$ and $\iota(w(v'))$, respectively, hence are subsets of A. It follows in particular that the links $\widehat{w}(U_{\ell(v_1)} \cap [v_1, v_2])$ and $\widehat{w}(U_{\ell(v')} \cap [v_1, v_2])$ both meet the arc $A \cap [\iota(\sigma), \iota(\tau)]_{T_0}$, which implies each link $\widehat{w}(U_\ell \cap [v_1, v_2])$, $\ell(v_1) < \ell < \ell(v')$, must meet A as well. But $\widehat{w}(U_{\ell(v)})$ has diameter $< 1/2$ and contains $\iota(w(v))$, hence misses A by the above. This is a contradiction, therefore v_1 and v_2 are $\leq$-adjacent.

(C3): Suppose $v \in V(G)$, v_1, v_2, v_3 are consecutive in G, and that $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0, 1]$ are such that $t' \geq t$, $w(v) = b_{t'}$, $v_1v_2v_3 \overset{w}{\mapsto} \sigma b_t \tau$, and $v_1 < v_2 \simeq v < v_3$. From Property (2) we know that $\ell(v)$ is between $\ell(v_1)$ and $\ell(v_3)$, hence the link $U_{\ell(v)}$ contains v and meets the arc $[v_1, v_2] \cup [v_2, v_3]$.

Since $d(\iota(b_{t'}), [\iota(\sigma), \iota(b_t)]_{T_0} \cup [\iota(b_t), \iota(\tau)]_{T_0}) = d(\iota(b_{t'}), \iota(b_t)) = t' - t$ and $\mathcal{U}$ has mesh $< 1/2 - \varepsilon$, it follows that $t' - t < 1/2$. ∎

4. Combinatorics of the graph-word ρ_N

4.1. Chain quasi-orders and ρ_N. Throughout this subsection assume that $\langle G, w\rangle$ is a compliant graph-word, and that $\leq$ is a chain quasi-order of $\langle G, w\rangle$.

Let $f : V(G) \to \mathbb{Z}$ be an order preserving function whose range is a contiguous block of integers.

LEMMA 6. *Suppose $v_1, \ldots, v_n$ are consecutive in G, and that for each $1 < j < n$ we have $w(v_{j-1}) \not\approx_\Gamma w(v_{j+1})$. Then $f(v_1), \ldots, f(v_n)$ are consecutive integers, i.e. either $f(v_{j+1}) = f(v_j) + 1$ for each $1 \leq j < n$, or $f(v_{j+1}) = f(v_j) - 1$ for each $1 \leq j < n$.*

Proof. This follows immediately from properties (C1) and (C2) of the chain quasi-order $\leq$. ∎

As an application of Lemma 6, we make the following observation.

LEMMA 7. *Suppose for some $i < 2N$ that $v_0, v_1, \ldots, v_{2\theta(i)} \in V(G)$ are consecutive in G with $v_0 \cdots v_{2\theta(i)} \overset{w}{\mapsto} \alpha_N(n(i)) \cdots \alpha_N(n(i+1))$. Let $k := f(v_0)$. Then we have one of the following four cases:*

(1) $v_0 \cdots v_{2\theta(i)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i))$;

(2) $v_0 \cdots v_{\theta(i)} \overset{f}{\mapsto} k \cdots (k + \theta(i))$, $v_{\theta(i)} \cdots v_{2\theta(i)} \overset{f}{\mapsto} (k + \theta(i)) \cdots k$;

(3) $v_0 \cdots v_{\theta(i)} \overset{f}{\mapsto} k \cdots (k - \theta(i))$, $v_{\theta(i)} \cdots v_{2\theta(i)} \overset{f}{\mapsto} (k - \theta(i)) \cdots k$; *or*

(4) $v_0 \cdots v_{2\theta(i)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i))$.

Moreover, the analogous statement holds for the word β_N^- (where we replace n with m and θ with ϕ).

Proof. This is a simple consequence of Lemma 6. ∎

LEMMA 8. *Suppose that for each $i \in \{0, N, 2N\}$, there are $v_1^{(i)}, v_2^{(i)}, v_3^{(i)} \in V(G)$ which are consecutive in G with $v_1^{(i)} v_2^{(i)} v_3^{(i)} \overset{w}{\mapsto} ab_{i/2N}c$. Then it cannot be the case that $v_3^{(0)} \simeq v_3^{(N)} \simeq v_3^{(2N)}$.*

Proof. Suppose for a contradiction that $f(v_3^{(0)}) = f(v_3^{(N)}) = f(v_3^{(2N)}) = k$. By Lemma 6, for each $i \in \{0, N, 2N\}$ we have

$$\text{either} \quad v_1^{(i)} v_2^{(i)} v_3^{(i)} \overset{f}{\mapsto} (k-2)\,(k-1)\,k \quad \text{or} \quad v_1^{(i)} v_2^{(i)} v_3^{(i)} \overset{f}{\mapsto} (k+2)\,(k+1)\,k.$$

It then follows from the pigeonhole principle that $f(v_2^{(i)}) = f(v_2^{(j)})$ for distinct $i, j \in \{0, N, 2N\}$. But this contradicts property (C3) of the chain quasi-order $\leq$. ∎

LEMMA 9. *Suppose $v_0, \dots, v_{|\alpha_N|-1} \in V(G)$ are consecutive in G and $v'_0, \dots, v'_{|\beta_N|-2} \in V(G)$ are consecutive in G with $v_0 \cdots v_{|\alpha_N|-1} \overset{w}{\mapsto} \alpha_N$ and $v'_0 \cdots v'_{|\beta_N|-2} \overset{w}{\mapsto} \beta_N^-$. Suppose further that $v_0 \simeq v'_0$. Then $v_1 \not\simeq v'_1$.*

Proof. Assume without loss of generality that $v_0 \leq v_1$. Suppose for a contradiction that $v_1 \simeq v'_1$.

We know that $f(v_0) \leq f(v_1)$ and that $f(v_0) = f(v'_0)$, $f(v_1) = f(v'_1)$. Put $k := f(v_{n(0)})$, and recall that $n(0) = 6N + 5 = m(0)$. It follows from Lemma 6 that

$$v_0 \cdots v_{n(0)} \overset{f}{\mapsto} (k - 6N - 5) \cdots k \quad \text{and} \quad v'_0 \cdots v'_{m(0)} \overset{f}{\mapsto} (k - 6N - 5) \cdots k.$$

CLAIM 9.1. *Let $i < 2N$. If $f(v_{n(i)}) = k$ and $f(v_{n(i)+\theta(i)}) < k$, then $f(v_{n(i+1)}) = k$. Similarly, if $f(v'_{m(i)}) = k$ and $f(v'_{m(i)+\phi(i)}) < k$, then $f(v'_{m(i+1)}) = k$.*

Proof. Suppose $f(v_{n(i)}) = k > f(v_{n(i)+\theta(i)})$. If

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i)),$$

then in particular $f(v_{n(i)+\theta(i)+1}) = k - \theta(i) - 1$. Also, $f(v_{n(0)-\theta(i)-1}) = k - \theta(i) - 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 7 we must have $f(v_{n(i+1)}) = k$.

Similarly, suppose $f(v'_{m(i)}) = k > f(v'_{m(i)+\phi(i)})$. If

$$v'_{m(i)} \cdots v'_{m(i+1)} \overset{f}{\mapsto} k \cdots (k - 2\phi(i)),$$

then in particular $f(v'_{m(i)+\phi(i)+1}) = k - \phi(i) - 1$. Also, $f(v'_{m(0)-\phi(i)-1}) = k - \phi(i) - 1$. But $w(v_{m(i)'+\phi(i)+1}) = b \not\approx_\Gamma c = w(v'_{m(0)-\phi(i)-1})$, so this contradicts (C1). Therefore by Lemma 7 we must have $f(v'_{m(i+1)}) = k$. ■(Claim 9.1)

CLAIM 9.2. *Either $f(v_{n(i)}) = k$ for each $i \leq 2N$ or $f(v'_{m(i)}) = k$ for each $i \leq 2N$.*

Proof. If $f(v_{n(i)+\theta(i)}) < k$ and $f(v'_{m(i)+\phi(i)}) < k$ for each $i < 2N$, then this follows immediately from Claim 9.1 and induction. Hence assume this is not the case, and let i^* be the smallest i for which $f(v_{n(i)+\theta(i)}) > k$ or $f(v'_{m(i)+\phi(i)}) > k$.

Observe that by Claim 9.1 and induction, we have $f(v_{n(i)}) = f(v'_{m(i)}) = k$ for each $i \leq i^*$.

CASE 1: $f(v'_{m(i^*)+\phi(i^*)}) > k$. It follows from Lemma 6 that

$$v'_{m(i^*)} \cdots v'_{m(i^*)+\phi(i')} \overset{f}{\mapsto} k \cdots (k + \phi(i^*)).$$

Suppose $i^* \le i < 2N$, and that $f(v_{n(i)}) = k$. If $f(v_{n(i)+\theta(i)}) < k$, then we deduce by Claim 9.1 that $f(v_{n(i+1)}) = k$.

If $f(v_{n(i)+\theta(i)}) > k$, suppose for a contradiction that

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i)).$$

In particular, this means $f(v_{n(i)+\theta(i)+1}) = k + \theta(i) + 1$. Also, since $\phi(i^*) > \theta(i)$, we have $f(v'_{m(i^*)+\theta(i)+1}) = k + \theta(i) + 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v'_{m(i^*)+\theta(i)+1})$, so this contradicts property (C1) of the chain quasi-order $\le$. Therefore by Lemma 7 we must have $f(v_{n(i+1)}) = k$.

Thus by induction, $f(v_{n(i)}) = k$ for each $i \le 2N$.

Case 2: $f(v'_{m(i^*)+\phi(i^*)}) < k$ and $f(v_{n(i^*)+\theta(i^*)}) > k$. Here we see by Claim 9.1 that $f(v'_{m(i^*+1)}) = k$. It follows from Lemma 6 that

$$v_{n(i^*)} \cdots v_{n(i^*)+\theta(i^*)} \overset{f}{\mapsto} k \cdots (k + \theta(i^*)).$$

Suppose $i^* + 1 \le i < 2N$ and $f(v'_{m(i)}) = k$. If $f(v'_{m(i)+\phi(i)}) < k$, then we find by Claim 9.1 that $f(v'_{m(i+1)}) = k$.

If $f(v'_{m(i)+\phi(i)}) > k$, suppose for a contradiction that

$$v'_{m(i)} \cdots v'_{m(i+1)} \overset{f}{\mapsto} k \cdots (k + 2\phi(i)).$$

In particular, this means $f(v'_{m(i)+\phi(i)+1}) = k + \phi(i) + 1$. Also, since $\theta(i^*) > \phi(i)$, we have $f(v_{n(i^*)+\phi(i)+1}) = k + \phi(i) + 1$. But $w(v'_{m(i)+\phi(i)+1}) = b \not\approx_\Gamma c = w(v_{n(i^*)+\phi(i)+1})$, so this contradicts (C1). Therefore by Lemma 7 we must have $f(v'_{m(i+1)}) = k$.

Thus by induction, $f(v'_{m(i)}) = k$ for each $i \le 2N$. $\blacksquare_{\text{(Claim 9.2)}}$

It only remains to notice that Claim 9.2 contradicts Lemma 8. So we must have $v_1 \not\simeq v'_1$. $\blacksquare$

For convenience in later statements and arguments, we will use the following notation:

Definition. Given $\sigma \in \Gamma$, define the word $\zeta_N(\sigma)$ by

$$\zeta_N(\sigma) := \begin{cases} \alpha_N & \text{if } \sigma = a, \\ \beta_N & \text{if } \sigma = b, \\ \gamma_N & \text{if } \sigma = c, \\ \beta_N^- b_t & \text{if } \sigma = b_t \text{ (for some } t \in [0,1]). \end{cases}$$

Lemma 10. *Suppose $\sigma, \tau \in \Gamma$, $v_0, \ldots, v_\kappa \in V(G)$ are consecutive in G, and $v'_0, \ldots, v'_\lambda \in V(G)$ are consecutive in G, with*

$$v_0 \cdots v_\kappa \overset{w}{\mapsto} \zeta_N(\sigma) \quad \textit{and} \quad v'_0 \cdots v'_\lambda \overset{w}{\mapsto} \zeta_N(\tau).$$

Suppose further that $v_0 \simeq v'_0$ and $v_1 \simeq v'_1$. Then $\sigma \approx_\Gamma \tau$.

Proof. Suppose for a contradiction that $\sigma \not\approx_\Gamma \tau$. If $\sigma = a$ and $\tau \in \{b, b_t : t \in [0,1]\}$, or vice versa, then this contradicts Lemma 9. If one of them is c, say σ, then $w(v_1) = c$ while $w(v_1') = b \not\approx_\Gamma c$, so this contradicts property (C1) of the chain quasi-order $\leq$. ■

Proposition 11. *There is no chain quasi-order for ρ_N, for any N.*

Proof. Suppose for a contradiction that $\leq$ is a chain quasi-order for ρ_N. Observe that since $r, p_1, q_1 \in V(G_{\rho_N})$ are all adjacent to o in G_{ρ_N}, these three vertices are also adjacent to o in the $\leq$ order. Hence by the pigeonhole principle, some pair of them are in the relation $\simeq$. But this is a contradiction by Lemma 10. ■

Oversteegen and Tymchatyn exhibit in [17] for each $\delta > 0$ a 2-dimensional plane strip with span $< \delta$ which has no chain cover of mesh < 1. Repovš et al. modify this example in [21] to construct for each $\delta > 0$ a tree in the plane with span $< \delta$ which has no chain cover of mesh < 1. In both examples, the diameters of the continua converge to ∞ as $\delta \to 0$. We pause to point out that we have now obtained a bounded family of such examples.

Corollary 12. *There is a uniformly bounded sequence $\langle T_N \rangle_{N=1}^{\infty}$ of simple triods in $\mathbb{R}^2$ such that for each N, $\operatorname{span}(T_N) < 1/N$ and T_N has no chain cover of mesh $< 1/4$.*

Proof. This is simply a combination of Propositions 1 (using T_0 and taking $\varepsilon \leq 1/2N$), 3, 5, and 11. ■

We are working to prove a stronger result: that there is a continuum in $\mathbb{R}^2$ which has span zero and cannot be covered by a chain of mesh less than some positive constant. To this end we will need some further technical combinatorial lemmas.

Lemma 13. *Suppose $\sigma, \tau \in \Gamma$ with $\sigma \approx_\Gamma \tau$, and that $v_0, \ldots, v_\kappa \in V(G)$ are consecutive in G and $v_0', \ldots, v_\kappa' \in V(G)$ are consecutive in G with*

$$v_0 \cdots v_\kappa \overset{w}{\mapsto} \zeta_N(\sigma) \quad \textit{and} \quad v_0' \cdots v_\kappa' \overset{w}{\mapsto} \zeta_N(\tau).$$

Then:

(i) *if $v_0 < v_1$, then $v_0 < v_j < v_\kappa$ for each $0 < j < \kappa$;*
(ii) *if $v_{\kappa-1} < v_\kappa$, then $v_0 < v_j < v_\kappa$ for each $0 < j < \kappa$;*
(iii) *if $v_0 \simeq v_0'$ and $v_1 \simeq v_1'$, then $v_\kappa \simeq v_\kappa'$;*
(iv) *if $v_\kappa \simeq v_\kappa'$ and $v_{\kappa-1} \simeq v_{\kappa-1}'$, then $v_0 \simeq v_0'$.*

Proof. Each of these statements is trivial if $\sigma = \tau = c$. We will prove the lemma for $\sigma = \tau = a$; the case $\sigma \approx_\Gamma \tau \approx_\Gamma b$ is handled analogously.

(i) Suppose $v_0 < v_1$.

Claim 13.1. $v_0 \cdots v_{n(0)} \overset{f}{\mapsto} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)})$.

Proof. This is immediate from Lemma 6. $\blacksquare_{\text{(Claim 13.1)}}$

CLAIM 13.2. *For each $i < 2N$, $v_{n(i)} \le v_{n(i+1)}$.*

Proof. We proceed by induction on $i < 2N$. Suppose the claim is true for each i' with $i' < i$. Put $k := f(v_{n(i)})$. Suppose for a contradiction that $f(v_{n(i)}) > f(v_{n(i+1)})$. By Lemma 7, this means

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i)).$$

In particular, we have $f(v_{n(i)+\theta(i)+1}) = k - \theta(i) - 1$.

Let j^* be the smallest $j \le i$ such that $f(v_{n(j)}) = k$.

If $j^* = 0$, then since $n(0) > \theta(i)$, we have $f(v_{n(0)-\theta(i)-1}) = k - \theta(i) - 1$. But also $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\le$.

If $j^* > 0$, then we know by Lemma 7 that

$$v_{n(j^*-1)} \cdots v_{n(j^*)} \overset{f}{\mapsto} (k - 2\theta(j^* - 1)) \cdots k.$$

Then we similarly observe that since $\theta(j^*-1) > \theta(i)$, we have $f(v_{n(j^*)-\theta(i)-1}) = k - \theta(i) - 1$. But also $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(j^*)-\theta(i)-1})$, so this contradicts (C1). $\blacksquare_{\text{(Claim 13.2)}}$

CLAIM 13.3. $v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5)$.

Proof. By Lemma 8 and Claim 13.2, we must have $v_{n(i-1)} < v_{n(i)}$ for some $0 < i \le 2N$; let i^* be the largest such i, so that $f(v_{n(2N)}) = f(v_{n(i^*)})$.

Suppose for a contradiction that

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) - 6N - 5).$$

Then in particular, since $6N + 5 > \theta(i^* - 1)$, we have $f(v_{n(2N)+\theta(i^*-1)+1}) = f(v_{n(2N)}) - \theta(i^*-1) - 1$. But also $f(v_{n(i^*)-\theta(i^*-1)-1}) = f(v_{n(2N)}) - \theta(i^*-1) - 1$ and $w(v_{n(i^*)-\theta(i^*-1)-1}) = c \not\approx_\Gamma a = w(v_{n(2N)+\theta(i^*-1)+1})$, so this contradicts (C1). Therefore by Lemma 6, we must have

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5). \ \blacksquare_{\text{(Claim 13.3)}}$$

It is now easy to check that $f(v_0) = f(v_{n(0)}) - 6N - 5 < f(v_j) < f(v_{n(2N)}) + 6N + 5 = f(v_\kappa)$ for any $0 < j < \kappa$.

(ii) Observe that if we consider the reverse order of $\le$, part (i) shows that if $v_0 > v_1$, then $v_0 > v_j > v_\kappa$ for each $0 < j < \kappa$. In particular, this would mean $v_{\kappa-1} > v_\kappa$. Therefore if $v_{\kappa-1} < v_\kappa$ then $v_0 < v_1$, hence the conclusion follows from part (i).

(iii) Suppose $v_0 \simeq v_0'$, $v_1 \simeq v_1'$, and assume without loss of generality that $v_0 < v_1$. This means Claims 13.1–13.3 hold for the v_j's and the v_j''s.

By Claim 13.1, we have

$$v_0 \cdots v_{n(0)} \overset{f}{\mapsto} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)}),$$
$$v'_0 \cdots v'_{n(0)} \overset{f}{\mapsto} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)}).$$

Claim 13.4. *For each $i \leq 2N$, $v_{n(i)} \simeq v'_{n(i)}$.*

Proof. Suppose not, and let i^* be the smallest $i < 2N$ such that $v_{n(i+1)} \not\simeq v'_{n(i+1)}$. Put $k := f(v_{n(i^*)}) = f(v'_{n(i^*)})$. It follows from Lemma 7 and Claim 13.2 that either $f(v_{n(i^*+1)}) = k$ and $f(v'_{n(i^*+1)}) > k$, or $f(v_{n(i^*+1)}) > k$ and $f(v'_{n(i^*+1)}) = k$; assume the former. This implies by Lemma 7 that

$$v'_{n(i^*)} \cdots v'_{n(i^*+1)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i^*)).$$

We claim that $f(v_{n(i)}) = k$ for each $i \geq i^*$. Indeed, given $i > i^*$, suppose for a contradiction that

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i)).$$

This means in particular that $f(v_{n(i)+\theta(i)+1}) = k + \theta(i) + 1$. Since $\theta(i) < \theta(i^*)$, we have $f(v'_{n(i^*)+\theta(i)+1}) = k + \theta(i) + 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v'_{n(i^*)+\theta(i)+1})$, so this contradicts (C1). Therefore, by Lemma 7 and Claim 13.2, we must have $f(v_{n(i+1)}) = k$. Hence, by induction, $f(v_{n(i)}) = k$ for each $i \geq i^*$.

In particular, $f(v_{n(2N)}) = k$. By Claim 13.3, we have

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} k \cdots (k + 6N + 5).$$

Since $6N + 5 > \theta(i^*)$, this means that $f(v_{n(2N)+\theta(i^*)+1}) = k + \theta(i^*) + 1$. Note $f(v'_{n(i^*)+\theta(i^*)+1}) = k + \theta(i^*) + 1$ as well. But $w(v'_{n(i^*)+\theta(i^*)+1}) = c \not\approx_\Gamma a = w(v_{n(2N)+\theta(i^*)+1})$, so this contradicts (C1). ■(Claim 13.4)

Claim 13.4 implies in particular that $f(v_{n(2N)}) = f(v'_{n(2N)})$. Then by Claim 13.3, we have

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5),$$
$$v'_{n(2N)} \cdots v'_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5).$$

This establishes part (iii).

(iv) Suppose $v_\kappa \simeq v'_\kappa$, $v_{\kappa-1} \simeq v'_{\kappa-1}$, and assume without loss of generality that $v_{\kappa-1} < v_\kappa$. By part (ii) this implies $v_0 < v_1$ and $v'_0 < v'_1$, so again Claims 13.1–13.3 hold for the v_j's and the v'_j's. By Claim 13.3, we have

$$v_\kappa \cdots v_{n(2N)} \overset{f}{\mapsto} (f(v_{n(2N)}) + 6N + 5) \cdots f(v_{n(2N)}),$$
$$v'_\kappa \cdots v'_{n(2N)} \overset{f}{\mapsto} (f(v_{n(2N)}) + 6N + 5) \cdots f(v_{n(2N)}).$$

Claim 13.5. *For each $i \le 2N$, $v_{n(i)} \simeq v'_{n(i)}$.*

Proof. Suppose not, and let i^* be the largest $i < 2N$ such that $v_{n(i)} \not\simeq v'_{n(i)}$. Put $k := f(v_{n(i^*+1)}) = f(v'_{n(i^*+1)})$. It follows from Lemma 7 and Claim 13.2 that either $f(v_{n(i^*)}) = k$ and $f(v'_{n(i^*)}) < k$, or $f(v_{n(i^*)}) < k$ and $f(v'_{n(i^*)}) = k$; assume the former. This implies by Lemma 7 that

$$v'_{n(i^*+1)} \cdots v'_{n(i^*)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i^*)).$$

We claim that $f(v_{n(i)}) = k$ for each $i \le i^*$. Indeed, given $i < i^*$, suppose for a contradiction that

$$v_{n(i+1)} \cdots v_{n(i)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i)).$$

Since $\theta(i^*) < \theta(i)$, this means in particular that $f(v_{n(i+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$. Note that $f(v'_{n(i^*+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$ as well. But $w(v_{n(i^*+1)-\theta(i^*)-1}) = c \not\approx_\Gamma a = w(v'_{n(i+1)-\theta(i^*)-1})$, so this contradicts (C1). Therefore by Lemma 7 and Claim 13.2 we must have $f(v_{n(i)}) = k$. Hence, by induction, $f(v_{n(i)}) = k$ for each $i \le i^*$.

In particular, $f(v_{n(0)}) = k$. By Claim 13.1, we have

$$v_{n(0)} \cdots v_0 \overset{f}{\mapsto} k \cdots (k - 6N - 5).$$

Since $6N + 5 > \theta(i^*)$, this means that $f(v_{n(0)-\theta(i^*)-1}) = k - \theta(i^*) - 1$. Note $f(v'_{n(i^*+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$ as well. But $w(v'_{n(i^*+1)-\theta(i^*)-1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i^*)-1})$, so this contradicts (C1). $\blacksquare_{\text{(Claim13.5)}}$

Claim 13.5 implies in particular that $f(v_{n(0)}) = f(v'_{n(0)})$. Then by Claim 13.1, we have

$$v_{n(0)} \cdots v_0 \overset{f}{\mapsto} f(v_{n(0)}) \cdots (f(v_{n(0)}) - 6N - 5),$$
$$v'_{n(0)} \cdots v'_0 \overset{f}{\mapsto} f(v_{n(0)}) \cdots (f(v_{n(0)}) - 6N - 5).$$

This establishes part (iv). $\blacksquare$

4.2. Iterated sketches. If $\iota_T : \Gamma \to T$ is a Γ-marking of the simple triod T and ρ_N is a $\langle T, \varepsilon\rangle$-sketch of the simple triod graph $T' := G_{\rho_N}$ such that $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T'} = [\iota_T(c), \iota_T(b)]_T$ (as in Proposition 1), then one can define an induced Γ-marking $\iota_{T'} : \Gamma \to T'$ on T' as follows: set $\iota_{T'}(a) := p_{|\alpha_N|-1}$, $\iota_{T'}(b) := q_{|\beta_N|-1} = \iota_T(b)$, $\iota_{T'}(c) := r$, and for each $t \in [0,1]$ put $\iota_{T'}(b_t) := \iota_T(b_t) \in [q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T'} = [\iota_T(c), \iota_T(b)]_T$.

Now let T_0 be as before, and suppose T_1 and T_2 are simple triods such that ρ_1 is a $\langle T_0, \varepsilon_0\rangle$-sketch of T_1, and ρ_2 is a $\langle T_1, \varepsilon_1\rangle$-sketch of T_2 (using the induced Γ-marking on T_1). Evidently we should be able to find a $\langle T_0, \varepsilon_0+\varepsilon_1\rangle$-sketch of T_2, and indeed this is necessary if we want to apply Proposition 5 to argue that T_2 has no chain cover of small mesh. This is the motivation for the next definition (see Proposition 14).

Definition. Suppose $\langle G, w\rangle$ is a compliant graph-word, and $N > 0$. A graph-word $\langle G^+, w^+\rangle$ is a *ρ_N-expansion of $\langle G, w\rangle$* if:

- G^+ is identical to G as a topological space, but the vertex set of G^+ is finer: for any adjacent pair of vertices $v_1, v_2 \in V(G)$, there are distinct degree 2 vertices $s_j^{v_1v_2}$, $j = 1, \ldots, \kappa_{v_1v_2}$, where $\kappa_{v_1v_2} = |\zeta_N(w(v_1))| + |\zeta_N(w(v_2))| - 3$, inserted into the edge joining v_1, v_2 so that $v_1, s_1^{v_1v_2}, \ldots, s_{\kappa_{v_1v_2}}^{v_1v_2}, v_2$ are consecutive in G^+;
- w^+ is defined by
$$v_1 s_1^{v_1v_2} \cdots s_{\kappa_{v_1v_2}}^{v_1v_2} v_2 \overset{w^+}{\longmapsto} \zeta_N(w(v_1))^{\leftarrow} \pitchfork \zeta_N(w(v_2))$$
when $v_1, v_2 \in V(G)$ are adjacent in G.

Remarks.

(1) Notice that $w^+|_{V(G)} = w$, and that $\langle G^+, w^+\rangle$ is also a compliant graph-word.

(2) Combinatorially, there is only one ρ_N-expansion of a given graph-word $\langle G, w\rangle$; however, geometrically they may differ according to where along the edges of G the extra vertices are inserted (though their order on the edge is uniquely determined by the definition).

Proposition 14. *Let T be a $\mathit{\Gamma}$-marked simple triod. Suppose ρ_N is a $\langle T, \varepsilon_1\rangle$-sketch of $T' := G_{\rho_N}$, and that $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T'} = [\iota_T(c), \iota_T(b)]_T$ (as in Proposition 1). Endow T' with a $\mathit{\Gamma}$-marking as at the beginning of Subsection 4.2. If $\rho = \langle G, w\rangle$ is a compliant graph-word which is a $\langle T', \varepsilon_2\rangle$-sketch of G, then there is a ρ_N-expansion of $\langle G, w\rangle$ which is a $\langle T, \varepsilon_1 + \varepsilon_2\rangle$-sketch of G.*

Proof. Let $\widehat{w_{\rho_N}} : T' \to T$ be a ρ_N-suggested bonding map such that $d(x, \widehat{w_{\rho_N}}(x)) < \varepsilon_1/2$ for each $x \in T'$, and let $\widehat{w} : G \to T'$ be a ρ-suggested bonding map such that $d(x, \widehat{w}(x)) < \varepsilon_2/2$ for each $x \in G$.

Consider any adjacent $v_1, v_2 \in V(G)$. Define
$$s_i^{v_1v_2} := \begin{cases} (\widehat{w}|_{[v_1,v_2]})^{-1}(p_{|\alpha_N|-1-i}) & \text{if } w(v_1) = a, \\ (\widehat{w}|_{[v_1,v_2]})^{-1}(q_{|\beta_N|-1-i}) & \text{if } w(v_1) \approx_{\mathit{\Gamma}} b, \\ (\widehat{w}|_{[v_1,v_2]})^{-1}(o) & \text{if } w(v_1) = c \end{cases}$$
for $1 \le i \le |\zeta_N(w(v_1))| - 1$, and
$$s_{\kappa_{v_1v_2}+1-i}^{v_1v_2} := \begin{cases} (\widehat{w}|_{[v_1,v_2]})^{-1}(p_{|\alpha_N|-1-i}) & \text{if } w(v_2) = a, \\ (\widehat{w}|_{[v_1,v_2]})^{-1}(q_{|\beta_N|-1-i}) & \text{if } w(v_2) \approx_{\mathit{\Gamma}} b \end{cases}$$
for $1 \le i \le |\zeta_N(w(v_2))| - 2$.

Let $V(G^+)$ be equal to $V(G)$ together with all these new vertices, and let w^+ be defined as in the definition of a ρ_N-expansion. Observe that $w^+ = w_{\rho_N} \circ (\widehat{w}|_{V(G^+)})$. Put $\rho^+ := \langle G^+, w^+ \rangle$, where G^+ is equal to G as a topological space, with vertex set $V(G^+)$.

It is now straightforward to see that $\widehat{w_{\rho_N}} \circ \widehat{w}$ is a ρ^+-suggested bonding map, and clearly $d(x, (\widehat{w_{\rho_N}} \circ \widehat{w})(x)) < (\varepsilon_1 + \varepsilon_2)/2$ for each $x \in G$. ∎

LEMMA 15. *Suppose $\langle G, w \rangle$ is a compliant graph-word, let $\langle G^+, w^+ \rangle$ be a ρ_N-expansion of $\langle G, w \rangle$, and suppose $\leq^+$ is a chain quasi-order of $\langle G^+, w^+ \rangle$.*

(i) *Let $v_1, v_2 \in V(G)$ be adjacent in G, and let $s_1, \ldots, s_\kappa \in V(G^+) \setminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2$ are consecutive in G^+. Then the following are equivalent:*

(1) *$v_1 <^+ v_2$;*
(2) *$v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$;*
(3) *$v_1 <^+ s_j <^+ v_2$ for some $j \in \{1, \ldots, \kappa\}$.*

(ii) *If $v_1, v_2 \in V(G)$ are adjacent in G and $v_1', v_2' \in V(G)$ are adjacent in G with $v_1 \simeq^+ v_1'$, $v_1 <^+ v_2$, and $v_1' <^+ v_2'$, then $v_2 \simeq^+ v_2'$.*

Proof. (i) The implications (2)⇒(3) and (3)⇒(1) are trivial. For (1)⇒(2) we will prove that $v_1 <^+ s_1$ implies that $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$. Then by considering the reverse order of $\leq^+$, it follows that $v_1 <^+ v_2$ implies $v_1 <^+ s_1$, hence $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$.

Suppose $v_1 <^+ s_1$. Let $i \in \{1, \ldots, \kappa\}$ be such that

$$s_i \cdots s_1 v_1 \overset{w^+}{\longmapsto} \zeta_N(w(v_1)) \quad \text{and} \quad s_i \cdots s_\kappa v_2 \overset{w^+}{\longmapsto} \zeta_N(w(v_2)).$$

By Lemma 13(ii), we have $v_1 <^+ s_j <^+ s_i$ for each $j \in \{1, \ldots, i-1\}$. Because G is compliant, we can deduce using Lemma 10 that $s_i <^+ s_{i+1}$. Then by Lemma 13(i) we have $s_i <^+ s_j <^+ v_2$ for each $j \in \{i+1, \ldots, \kappa\}$.

(ii) Suppose $v_1, v_2 \in V(G)$ are adjacent in G and $v_1', v_2' \in V(G)$ are adjacent in G with $v_1 \simeq^+ v_1'$, $v_1 <^+ v_2$, and $v_1' <^+ v_2'$. Let $s_1, \ldots, s_\kappa$ and i be as in part (i), and let $s_1', \ldots, s_\lambda' \in V(G^+) \setminus V(G)$ be such that $v_1', s_1', \ldots, s_\lambda', v_2'$ are consecutive in G^+ and

$$v_1' s_1' \cdots s_\lambda' v_2' \overset{w^+}{\longmapsto} \zeta_N(w(v_1'))^{\leftarrow} \pitchfork \zeta_N(w(v_2')).$$

By property (C1) of the chain quasi-order $\leq^+$, $w(v_1) \approx_\Gamma w(v_1')$, hence $|\zeta_N(v_1)| = |\zeta_N(v_1')|$, and so

$$s_i' \cdots s_1' v_1' \overset{w^+}{\longmapsto} \zeta_N(w(v_1')) \quad \text{and} \quad s_i' \cdots s_\lambda' v_2' \overset{w^+}{\longmapsto} \zeta_N(w(v_2')).$$

By Lemma 13(iv), we have $s_i \simeq^+ s_i'$, and as in part (i) we know that $s_{i+1}' >^+ s_i'$. By Lemma 10, this implies $w(v_2) \approx_\Gamma w(v_2')$, hence $\kappa = \lambda$. Then by Lemma 13(iii), we conclude that $v_2 \simeq^+ v_2'$. ∎

Proposition 16. *Suppose $\langle G, w\rangle$ is a compliant graph-word. If some (any) ρ_N-expansion of $\langle G, w\rangle$ has a chain quasi-order, then $\langle G, w\rangle$ also has a chain quasi-order.*

Proof. Let $\langle G^+, w^+\rangle$ be a ρ_N-expansion of $\langle G, w\rangle$, and let $\leq^+$ be a chain quasi-order of $\langle G^+, w^+\rangle$.

Define $\leq$ on $V(G)$ by $\leq \,:=\, \leq^+|_{V(G)}$. Clearly $\leq$ is a total quasi-order since $\leq^+$ is. We must check that $\leq$ has properties (C1)–(C3):

(C1): This is immediate since $\leq^+$ has this property.

(C2): We will need the following claim:

Claim 16.1. *In $\langle G^+, w^+\rangle$, if $v \in V(G)$ and $v' \in V(G^+)$ are such that $v \simeq^+ v'$, then in fact $v' \in V(G)$.*

Proof. We proceed by induction on the number of vertices in G.

If $|V(G)| = 1$, then there is nothing to prove.

Assume the claim holds for all graph-words whose graph has n or fewer vertices, and assume $|V(G)| = n+1$. Let $u \in V(G)$ be such that the subgraph G^- obtained by removing the vertex u (and all edges emanating from u) is connected. There is a ρ_N-expansion of $\langle G^-, w|_{V(G)\smallsetminus\{u\}}\rangle$ which is a sub-graph-word of $\langle G^+, w^+\rangle$ (it has vertex set $V(G^+) \cap G^-$), and the restriction of $\leq^+$ to this sub-graph-word is a chain quasi-order. By induction, the claim holds for G^-.

Let $u' \in V(G)\smallsetminus\{u\}$ be adjacent to u in G. Let $s_1, \dots, s_\kappa \in V(G^+)\smallsetminus V(G)$ be such that $u', s_1, \dots, s_\kappa, u$ are consecutive in G^+ and

$$u' s_1 \cdots s_\kappa u \overset{w^+}{\longmapsto} \zeta_N(w(u'))^{\leftarrow} \pitchfork \zeta_N(w(u)).$$

Assume $u' <^+ u$ (the other case is similar), which implies by Lemma 15(i) that $u' <^+ s_j <^+ u$ for each $j \in \{1, \dots, \kappa\}$.

We have four things to check:

1. for each $y \in V(G) \smallsetminus \{u\}$ and each $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, $y \not\simeq^+ s$;
2. for each $y \in V(G) \smallsetminus \{u\}$ and each $j \in \{1, \dots, \kappa\}$, $y \not\simeq^+ s_j$;
3. for each $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, $u \not\simeq^+ s$; and
4. for each $j \in \{1, \dots, \kappa\}$, $u \not\simeq^+ s_j$.

Observe that (1) holds by induction, and (4) is immediate from the observation that $u' <^+ s_j <^+ u$ for each $j \in \{1, \dots, \kappa\}$. For (2) and (3), we consider two cases.

Case 1: For every $y \in V(G) \smallsetminus \{u\}$, $y \leq^+ u'$. Since $u' <^+ s_j <^+ u$ for each $j \in \{1, \dots, \kappa\}$, we immediately see that $y \not\simeq^+ s_j$ for any $y \in V(G)\smallsetminus\{u\}$.

Also, from Lemma 15 (i) it follows that for every $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, $s <^+ u'$. Therefore $u \not\simeq^+ s$ for any such s.

Case 2: There exists some $y \in V(G) \smallsetminus \{u\}$ such that $u' <^+ y$. Let $\mathcal{P}$ be a path of vertices in G^- starting at u' and ending at y. Let y_1 be the latest vertex y' in $\mathcal{P}$ with $y' \leq^+ u'$, and let y_2 be the next vertex in $\mathcal{P}$ after y_1, so that y_1 and y_2 are adjacent in G and $y_1 \leq^+ u' <^+ y_2$.

Suppose for a contradiction that $y_1 <^+ u'$. Let $z_1, \ldots, z_\lambda \in V(G^+) \smallsetminus V(G)$ be such that $y_1, z_1, \ldots, z_\lambda, y_2$ are consecutive in G^+. Then by Lemma 4 there is some $i \in \{1, \ldots, \lambda\}$ such that $u' \simeq^+ z_i$. But the claim holds for G^- by induction, so this is a contradiction. Therefore we must have $u' \simeq^+ y_1$.

Then from Lemma 15(ii) we know that $u \simeq^+ y_2$. It follows immediately that $u \not\simeq^+ s$ for each $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, because $y_2 \not\simeq^+ s$ for every such s by induction.

Moreover, for each $j \in \{1, \ldots, \kappa\}$, since $y_1 \simeq^+ u' <^+ s_j <^+ u \simeq^+ y_2$, we know from Lemma 4 that there is some $s \in V(G^+) \smallsetminus V(G)$ inserted between y_1 and y_2 such that $s_j \simeq^+ s$. It follows that $y \not\simeq^+ s_j$ for any $y \in V(G) \smallsetminus \{u\}$, because $y \not\simeq^+ s$ for every such y by induction. ■(Claim 16.1)

Now suppose $v_1, v_2 \in V(G)$ are adjacent in G, and assume $v_1 \leq v_2$. Let $s_1, \ldots, s_\kappa \in V(G^+) \smallsetminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2$ are consecutive in $V(G^+)$. If $v \in V(G)$ were such that $v_1 < v < v_2$, then $v_1 <^+ v <^+ v_2$ as well, so by Lemma 4 there would be some $i \in \{1, \ldots, \kappa\}$ such that $v \simeq^+ s_i$. But this contradicts Claim 16.1.

(C3): Suppose $v \in V(G)$, v_1, v_2, v_3 are consecutive in G, and $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0,1]$ are such that $t' \geq t$, $w(v) = b_{t'}$, $v_1 v_2 v_3 \overset{w}{\mapsto} \sigma b_t \tau$, and $v_1 < v_2 \simeq v < v_3$.

Let $s_1, \ldots, s_\kappa, s'_1, \ldots, s'_\lambda \in V(G^+) \smallsetminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2, s'_\lambda, \ldots, s'_1, v_3$ are consecutive in G^+, and

$$v_1 s_1 \cdots s_\kappa v_2 s'_\lambda \cdots s'_1 v_3 \overset{w^+}{\longmapsto} \zeta_N(\sigma)^{\leftarrow} \pitchfork \beta_N^- b_t (\beta_N^-)^{\leftarrow} \pitchfork \zeta_N(\tau).$$

Observe that $w^+(s_\kappa) = w^+(s'_\lambda) = c$. Since $v_1 <^+ v_2$, by Lemma 15(i) we must have $s_\kappa <^+ v_2$. Likewise, we have $v_2 <^+ s'_\lambda$. It now follows from property (C3) of the chain quasi-order $\leq^+$ that $t' - t < 1/2$. ■

5. The example

Example. *There exists a continuum $X \subset \mathbb{R}^2$ which is non-chainable and has span zero.*

Proof. First we define by recursion a sequence $\langle T_N \rangle_{N=0}^{\infty}$ of Γ-marked simple triods in $\mathbb{R}^2$ and a sequence $\langle \varepsilon_N \rangle_{N=0}^{\infty}$ of positive real numbers as follows.

Let $T_0 \subset \mathbb{R}^2$ be as defined in Subsection 2.1, and put $\varepsilon_0 := 1/8$.

Suppose T_N, ε_N have been defined. Apply Proposition 1 to obtain an embedding T_{N+1} of the simple triod graph $G_{\rho_{N+1}}$ in $\mathbb{R}^2$ such that ρ_{N+1} is a $\langle T_N, \varepsilon_N \rangle$-sketch of T_{N+1}, and $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T_{N+1}} = [\iota_{T_N}(c), \iota_{T_N}(b)]_{T_N}$. Endow T_{N+1} with a Γ-marking as that at the beginning of Subsection 4.2. Notice that $T_{N+1} \subset (T_N)_{\varepsilon_N}$, where Y_ε denotes the ε-neighborhood of the space Y in $\mathbb{R}^2$. By Proposition 3, the span of T_{N+1} is $< 1/2(N+1) + \varepsilon_N$. Let $0 < \varepsilon_{N+1} < 2^{-N-4}$ be small enough so that $\overline{(T_{N+1})_{\varepsilon_{N+1}}} \subseteq \overline{(T_N)_{\varepsilon_N}}$, and so that $\mathrm{span}(\overline{(T_{N+1})_{\varepsilon_{N+1}}}) < 1/2(N+1) + 2\varepsilon_N$.

Put $X := \bigcap_{N=0}^{\infty} \overline{(T_N)_{\varepsilon_N}}$. Observe that for any N, we have $X \subseteq \overline{(T_{N+1})_{\varepsilon_{N+1}}}$, hence

$$\mathrm{span}(X) \leq \mathrm{span}(\overline{(T_{N+1})_{\varepsilon_{N+1}}}) < \frac{1}{2(N+1)} + 2\varepsilon_N.$$

Since ε_N converges to 0 as $N \to \infty$, it follows that X has span zero.

Suppose for a contradiction that X has a chain cover of mesh $< 1/4$. Then there is some $N > 0$ for which T_N has a chain cover of mesh $< 1/4$.

Define by recursion the graph-words $\langle G_i, w_i \rangle$, $0 \leq i \leq N-1$, as follows: $\langle G_{N-1}, w_{N-1} \rangle := \rho_N$, and for $i < N-1$, $\langle G_i, w_i \rangle$ is the ρ_{i+1}-expansion of $\langle G_{i+1}, w_{i+1} \rangle$ provided by Proposition 14 which is a $\langle T_i, \sum_{j=i}^{N-1} \varepsilon_j \rangle$-sketch of T_N. In particular, $\langle G_0, w_0 \rangle$ is a $\langle T_0, \sum_{j=0}^{N-1} \varepsilon_j \rangle$-sketch of T_N.

Since $\sum_{j=0}^{N-1} \varepsilon_j < \sum_{j=0}^{N-1} 2^{-j-3} < 1/4$, Proposition 5 shows that $\langle G_0, w_0 \rangle$ has a chain quasi-order. Then by Proposition 16 and induction, we obtain a chain quasi-order for each graph-word $\langle G_i, w_i \rangle$. In particular, $\langle G_{N-1}, w_{N-1} \rangle$ has a chain quasi-order. But $\langle G_{N-1}, w_{N-1} \rangle$ is ρ_N, so this contradicts Proposition 11. ∎

6. Questions. The construction presented in this paper can be carried out so that every proper subcontinuum of the resulting space is an arc; hence, in particular, it is far from being hereditarily indecomposable. On the other hand, it follows from results of [17] that if there exists a non-degenerate homogeneous continuum in the plane which is not homeomorphic to the circle, the pseudo-arc, or the circle of pseudo-arcs, then there would be one which is hereditarily indecomposable and with span zero. Given that the pseudo-arc is the only hereditarily indecomposable chainable continuum [2], this raises the following question:

Question 1 (see Problem 9 of [18]). *Is there a hereditarily indecomposable non-chainable continuum with span zero?*

If such an example exists, then by [19, Corollary 6] it would be a continuous image of the pseudo-arc. Since any map to a hereditarily indecomposable continuum is confluent [23, Lemma 15], it would also be a counterexample to Problem 84 of [5], which asks whether every confluent image of a chainable continuum is chainable.

Regarding the planarity of the example in this paper, while every chainable continuum can be embedded in the plane [2], the same is not known to be true of continua with span zero.

QUESTION 2. *Can every continuum with span zero be embedded in $\mathbb{R}^2$?*

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