1. A continuous function whose Fourier series diverges

When we were studying the convergence of Fourier series, we stated the following theorem.

**Theorem 1.** There is a continuous periodic function $f$ whose Fourier series diverges. In fact, the partial sums $S_N f$ obey

$$\sup_{N} |S_N f(0)| = \infty.$$ 

We did not prove Theorem 1, but we did prove the following lemma.

**Lemma 1.** For any $M > 0$, there is a continuous periodic function $g_M$ with $\|g_M\|_{C^0} \leq 1$ but

$$\sup_{N} |S_N g_M(0)| > M.$$ 

In this section, we explain how to get from Lemma 1 to Theorem 1. The argument that we use can be generalized and applied to other situations. In the next section, we look back at the argument from a more abstract point of view. We isolate the ingredients that we need to make the argument work and formulate it as a general theorem about maps between Banach spaces. This general theorem is called the Uniform Boundedness Principle. It was proven by Banach and Steinehaus in the 30’s.

**Proof of Theorem 1:**

Let $Bad(M)$ denote the set of continuous periodic functions $g$ with the property that

$$\sup_{N} |S_N g(0)| > M \|g\|_{C^0}.$$  

(1)

It follows from the definition that the sets $Bad(M)$ are nested: $Bad(1) \supset Bad(2) \supset Bad(3) \supset \ldots$. Also, we know from Lemma 1 that $Bad(M)$ is non-empty for every $M$. We would like to prove that the intersection of all the sets $Bad(M)$ is non-empty. Indeed, since $f \in \cap_{M=1}^\infty Bad(M)$. Then $f$ is a continuous function, and $\sup_{N} |S_N f(0)| = \infty$.

Now in general, it is possible to find an infinite sequences of non-empty nested sets $S(1) \supset S(2) \supset \ldots$ so that $\cap_M S(M)$ is empty. For example, we may have
\( S(M) = \{ x \in \mathbb{R} | x > M \} \). The sets \( \text{Bad}(M) \) have an important extra property which makes them nicer than general sets: the spaces \( \text{Bad}(M) \) are dense open subsets of a Banach space.

We let \( C^0_{\text{per}} \) denote the space of continuous periodic functions. We equip this space with the sup-norm. It is then a Banach space. The sets \( \text{Bad}(M) \) are subsets of \( C^0_{\text{per}} \). We will prove that each set \( \text{Bad}(M) \) is open and dense.

**Lemma 2.** Each set \( \text{Bad}(M) \) is open.

**Proof.** Suppose that \( g \) is in \( \text{Bad}(M) \). Then for some \( N \), \( |S_N g(0)| > M \| g \|_{C^0} \). We can choose \( \epsilon > 0 \) so that

\[
|S_N g(0)| > M \| g \|_{C^0} + \epsilon. \tag{1}
\]

We will prove that \( |S_N h(0)| > M \| h \|_{C^0} \) provided that \( \| g - h \|_{C^0} < \delta \).

For our fixed value of \( N \), the function \( S_N f(0) \) is a continuous function of \( f \in C^0_{\text{per}} \). In fact, if \( \| f_1 - f_2 \|_{C^0} < \delta \), then \( |\hat{f}_1(n) - \hat{f}_2(n)| < \delta \) for every \( n \). But \( S_N f(0) = \sum_{n=0}^{N} \hat{f}(n) \), and so \( |S_N f_1(0) - S_N f_2(0)| < (2N+1)\delta \). Hence \( |S_N h(0) - S_N g(0)| < \epsilon/2 \) provided that \( \| g - h \|_{C^0} < (10N)^{-1} \epsilon \).

Also, if \( \| g - h \|_{C^0} < (1/2)M^{-1} \epsilon \), then \( |M \| h \|_{C^0} - M \| g \|_{C^0} | < \epsilon/2 \).

We take \( \delta \) to be the minimum of the two numbers considered above. If \( \| g - h \|_{C^0} < \delta \), then replacing \( g \) by \( h \) changes each side of (1) by at most \( \epsilon/2 \), and so \( h \) is still in \( \text{Bad}(M) \). \( \square \)

**Lemma 3.** Each set \( \text{Bad}(M) \) is dense.

**Proof.** Let \( f \) be any continuous periodic function. We have to construct a sequence of functions \( f_i \in \text{Bad}(M) \) with \( f_i \to f \) in \( C^0 \).

If \( f \) is in \( \text{Bad}(M) \) then we take \( f_i = f \) for all \( i \). If not, then \( f \) obeys the following inequality for all \( N \geq 0 \):

\[
|S_N f(0)| \leq M \| f \|_{C^0}.
\]

Define \( f_i = f + 2^{-i}g_{4^i} \). Here the function \( g_{4^i} \) is one of the functions that appears in Lemma 1 above. It has \( \| g_{4^i} \|_{C^0} \leq 1 \), and so \( f_i \to f \). On the other hand,

\[
S_N f_i(0) = S_N f(0) + 2^{-i}S_N g_{4^i}(0).
\]

The first term \( S_N f(0) \) is bounded uniformly in \( N \). On the other hand, for each \( i \), we can choose \( N \) so that \( |S_N g_{4^i}(0)| > 4^i \). Therefore, for all sufficiently large \( i \), \( f_i \in \text{Bad}(M) \). \( \square \)

Once we know that each set \( \text{Bad}(M) \) is dense and open, there is a general approach to show that they all intersect. This argument is due to Baire, and it is called the Baire category theorem.
Baire Category Theorem. Let \( X \) be a complete metric space. Let \( O_i \) be dense open subsets of \( X \), where \( i = 1, 2, 3, \ldots \). Then the countable intersection \( \cap_{i=1}^{\infty} O_i \) is not empty.

Proof. Pick a point \( p_1 \) in \( O_1 \). Since \( O_1 \) is open, we can find a ball \( B_1 \subset O_1 \) where \( B_1 \) is centered at \( p_1 \). Since \( O_2 \) is dense, we can find a point \( p_2 \in O_2 \) which lies as close as we like to \( p_1 \). In particular, we can arrange that \( p_2 \) lies in the (open) ball \( B_1 \). Since \( O_2 \) is open, we can find a ball \( B_2 \), centered at \( p_2 \), with \( B_2 \subset O_2 \). Since \( p_2 \) is in the interior of \( B_1 \), we can choose the radius of \( B_2 \) small enough that the closure \( \overline{B_2} \subset B_1 \subset O_1 \). Proceeding similarly, we find a nested sequence of balls \( B_1 \supset B_2 \supset B_3 \supset \ldots \) with \( B_i \subset O_i \). At each stage, we choose \( B_i \) small enough that \( \overline{B_i} \subset B_{i-1} \). We can choose the radii as small as we like, so we can also assume that the radius of \( B_i \) is at most \( 2^{-i} \). Let \( p_i \) be the center of \( B_i \). If \( i, j > N \), then both \( p_i \) and \( p_j \) lie in the ball \( B_N \) with radius at most \( 2^{-N} \), and so the distance \( d(p_i, p_j) < 2 \cdot 2^{-N} \). Hence the sequence \( p_i \) is Cauchy. Because \( X \) is complete, \( p_i \) converges to a point \( p \). The point \( p \) lies in each set \( O_i \) for the following reason. For \( j \geq i + 1 \), we know that all \( p_j \) lie in \( B_{i+1} \). Hence the limit point \( p \) lies in \( \overline{B_{i+1}} \subset B_i \subset O_i \). □

2. The Uniform Boundedness Principle

The argument above is general, and it applies to many other problems. Next we isolate the ingredients that we needed to make the argument above work - formulating a general theorem about maps between Banach spaces.

Theorem 2. Let \( B_1, B_2 \) be Banach spaces. Let \( T_N \) be bounded linear maps from \( B_1 \) to \( B_2 \). Suppose that for each \( M > 0 \), there exists an element \( g_M \in B_1 \) so that

\[
\|g_M\|_{B_1} \leq 1, \quad \sup_{N} \|T_N g_M\|_{B_2} > M.
\]

Then there exists one single element \( f \in B_1 \) so that

\[
\sup_{N} \|T_N f\|_{B_2} = +\infty.
\]

Proof. The proof is essentially the same as the proof of Theorem 1. First we define sets \( \text{Bad}(M) \).

\[
\text{Bad}(M) := \left\{ g \in B_1 \text{ such that } \sup_{N} \|T_N g\|_{B_2} > M \|g\|_{B_1} \right\}.
\]

Lemma 4. For each \( M > 0 \), the set \( \text{Bad}(M) \) is open in \( B_1 \).

Proof. The proof is essentially the same as Lemma 2 above. I’m going to write it a little differently. First we observe that \( \text{Bad}(M) \) is an infinite union:

\[
\text{Bad}(M) := \bigcup_{N} \{ g \in B_1 \text{ such that } \|T_N g\|_{B_2} > M \|g\|_{B_1} \}.
\]
It suffices to show that each set \( \{ g \in B_1 \text{ such that } \| T_N g \|_{B_2} > M \| g \|_{B_1} \} \) is open. This follows since the function that sends \( g \) to \( \| T_N g \|_{B_2} - M \| g \|_{B_1} \) is continuous.

\[ \square \]

**Lemma 5.** Each set \( \text{Bad}(M) \) is dense in \( B_1 \).

This proof is essentially the same as Lemma 3. Fill in the details as an exercise.

By the Baire Category Theorem, there is an \( f \in B_1 \) lying in \( \cap_{M=1}^{\infty} \text{Bad}(M) \). Therefore, \( \sup_N \| T_N f \|_{B_1} = +\infty \).

\[ \square \]

For example, recall problem 2 of problem set 8.

2. For any number \( M > 0 \), find a function \( f_M \in L^1([0, 2\pi]) \) with \( \| f_M \|_{L^1} \leq 1 \) so that

\[ \sup_N \| S_N f_M \|_{L^1} > M. \]

Applying the uniform boundedness principle we get the following corollary.

**Corollary.** There exists a function \( f \in L^1([0, 2\pi]) \) so \( \sup_N \| S_N f \|_{L^1([0,2\pi])} = +\infty \). In particular, the partial sums \( S_N f \) do not converge to \( f \) in \( L^1 \).