

## 5. ABSOLUTE EXTREMA

### Definition, Existence & Calculation

We assume that the definition of function is known and proceed to define “absolute minimum”. We also assume that the student is familiar with the terms “domain” and “range” of a function.

**Definition.** A real valued function on a set  $S$  ( $f : S \rightarrow R$ , where  $R$  denotes the real numbers) has an absolute minimum at the point  $a \in S$  if  $f(a) \leq f(x)$  for all  $x \in S$ .

Similarly,  $f$  has an absolute maximum at  $b \in S$  if  $f(x) \leq f(b)$  for all  $x \in S$ .

If we want to refer to one of absolute maximum or minimum without specifying which one we call it an absolute extremum.

It is important to note that an absolute maximum requires that the largest value of  $f$  on the set is actually assumed at some point and that an absolute minimum requires that the smallest value of the function is actually assumed at some point.

### Examples:

The following examples are chosen to illustrate a range of manifestations of the concepts of absolute maximum and minimum.

(1) Let  $S$  be the set of students registered in MAT 133 on a given date. Let  $h(x)$  be the height of student  $x$  in  $cm$ . If  $a$  is the shortest student (or one of them, if there are several students who qualify for the shortest height) and  $b$  is the tallest student (or one of them if several students qualify for being tallest), then  $a$  and  $b$  are the absolute minimum and maximum points of the function  $h$ .

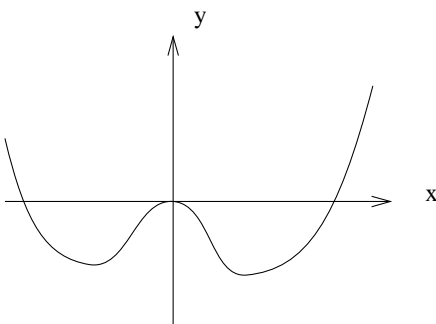
(2)  $S = \mathbb{R}; f(x) = x$ .  $f$  increases from  $-\infty$  to  $\infty$  and has no absolute extrema.

(3)  $S = \mathbb{R}; f(x) = 1$ . Every point is an absolute maximum and an absolute minimum.

(4)  $S = [0, 1]; f(x) = x$ .  $f$  increases from 0 to 1 and has absolute minimum 0 at  $x = 0$  and absolute maximum 1 at  $x = 1$ .

(5)  $S = \mathbb{R}^3; f(a, b, c) = a^2 + b^2 + c^2$ .  $f$  has absolute minimum 0 at the origin, no absolute maximum.

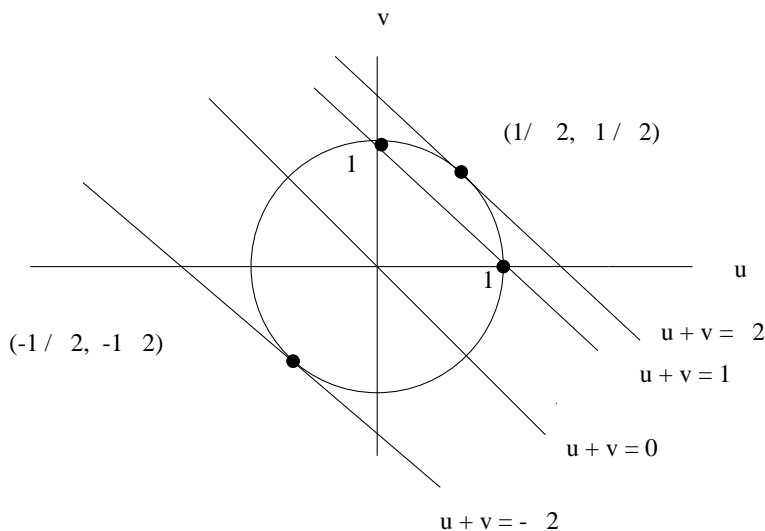
(6) In this example  $S$  is the set of real numbers  $R$ . Whenever  $S$  is a subset of  $R$ , graphs are very useful to help understand what is going on. Accordingly, let  $f : R \rightarrow R$  and  $f(x) = x^4 - 2x^2$ . The graph looks like



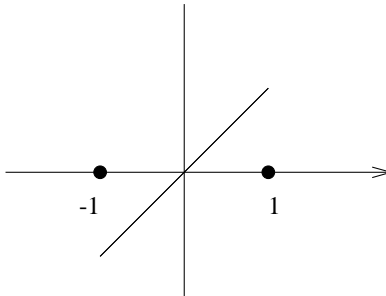
It is evident from the graph that the lowest points occur at  $(-1,-1)$  and  $(1,-1)$ . Consequently the points  $+1$  and  $-1$  are absolute minima of  $f$ . There is no highest point on the graph (it rises to  $\infty$ ) so there is no absolute maximum.

(7) In this example  $S$  is a subset of the plane which has been fitted with the usual coordinate system. In such examples a specialized graphical representation is often used for real functions. Let

$S = \{(u, v) \in \mathbb{R}^2 | u^2 + v^2 \leq 1\}$ ;  $f(u, v) = u + v$ . Here  $f$  has a single absolute maximum at  $(1/\sqrt{2}, 1/\sqrt{2})$  and a single absolute minimum at  $(-1/\sqrt{2}, -1/\sqrt{2})$ .



(8) Our final example is simple but important. It shows that a function can appear to have a largest and smallest value in its range and yet fail to have an absolute maximum or minimum because these values are not assumed for any points in the domain of the function. Let  $S = \{x : -1 < x < 1\}$  and  $f(x) = x$ . The graph of  $f$  looks as follows:



It is clear that  $-1 < f(x) < 1$ , but  $f$  never assumes the values  $-1$  or  $+1$  on  $S$ . Consequently  $f$  has no absolute minimum nor absolute maximum.

We are finished with examples for the moment but we proceed to explore the lesson of Example 8 with some care.

If a real valued function  $f$  is to have an absolute maximum on a set  $S$ , there must exist a number  $M$  which is greater than or equal to all the values assumed by  $f$  and there must be a point  $p$  within  $S$  for which  $f(p) = M$ . For some purposes it is useful to have a definition which embodies the first of the foregoing requirements.

**Definition.** A function is bounded above on a set  $S$  if there exists a number  $M$  for which  $s \in S$  implies  $f(s) \leq M$ .

Analogously, if a real valued function  $f$  is to have an absolute minimum on a set  $S$ , there must exist a number  $m$  which is less than or equal to all the values assumed by on points of  $S$  and a point  $p \in S$  where  $f(p) = m$ . We are thus led to define:

A function  $f$  is bounded below on  $S$  if there exists a number  $m$  for which  $s \in S$  implies  $m \leq f(s)$ .

Example (8) above shows that a function can be bounded above and below yet have no absolute maximum or minimum.

It would be useful to have concise conditions which guarantee the existence of an absolute maximum or minimum. Such conditions are known. They involve a condition to be satisfied by the function and a condition to be satisfied by the set. More precisely, the result is:

**Extreme Value Theorem:**

**A continuous function on a closed bounded set has an absolute maximum and minimum.**

The extreme value theorem is covered in the text Haeussler and Paul in section 14.2. We shall add to that discussion.

Although the extreme value theorem is a very general result which applies to real valued functions with a domain in any number of dimensions, we shall be concerned with only the simplest instance of a real valued function on a subset of the real numbers. We are not able to rigorously prove the result because we do not have a sufficiently complete description of the real numbers (in particular, we lack something called the least upper bound axiom). Instead we shall be satisfied with explaining what the result means and learning how to use it.

We already have considerable experience and understanding of continuity since it has been covered in the main text for the course. The remaining terms in the hypothesis of the theorem are “closed bounded set”.

### Meaning of BOUNDED

A set of  $S$  real numbers is said to be bounded if there exists a number  $L$  which is less than all the numbers in  $S$  and another number  $U$  which is greater than all the numbers in  $S$ . that is  $s \in S \Rightarrow L \leq s \leq U$  (where “ $\Rightarrow$ ” means “implies”, as usual).

For example, the set of all positive real numbers is not bounded because there is no upper bound  $U$ . whatever number one might try for  $U$ , one will find a positive number which is bigger than  $U$ . On the other hand, any negative number will do for  $L$ . Note that the set  $S = \{x : 1 < x < 5\}$  is bounded and there are many choices for  $L$  and  $U$  ( $L = 0$ ,  $U = 10$  will do just fine).

### Meaning of CLOSED

A set  $S$  of real numbers is said to be closed if it contains all its boundary points. A boundary point  $b$  of  $S$  is defined as a point (not necessarily a member of  $S$ ) which has the property that every interval with centre  $b$  contains both a point from  $S$  and a point from outside  $S$ . A little thought should convince the reader that this definition of boundary point gives rise to a boundary consistent with our intuitive understanding of boundary. Under this definition, it turns out that the set of real numbers is closed because there are no boundary points so that the containment requirement is trivially satisfied. Apart from this example we need to understand that an interval  $\{x : a < x \leq b\}$ , commonly written  $(a, b]$ , is not closed because it does not contain its left boundary point. However, the set  $\{x : a \leq x \leq b\}$ , commonly written  $[a, b]$ , is closed because it contains both its boundary points.

### Extreme Value Theorem - Discussion

The definitions of continuity and closed bounded set are not quite sufficient to enable us

to prove this theorem rigorously. The stumbling block is a shortcoming in the definition of real numbers as commonly understood at this stage. The real numbers possess a property known as the “least upper bound axiom” which can be stated simply enough as “every bounded set of real numbers has a least upper bound”. However, the technical foundation needed to work with the least upper bound axiom is beyond the scope of this course.

The least upper bound axiom is used to prove that every continuous function on a closed bounded set is bounded above and below. That is, the set of values assumed by the function (actual range) is bounded above and below. This is the hard part.

The easy part is to show that the function takes on the value of the least upper bound of its actual range. This is how the argument goes. Suppose that  $M$  is the least upper bound of its range but no  $x$  exists for which  $f(x) = M$ . Then consider the function  $g(x) = 1/(M - f(x))$ . The function  $g$  is again continuous on the same domain as  $f$ . Consequently, it must be bounded above - by a number  $B$ , say, that is

$$\frac{1}{M - f(x)} \leq B.$$

It follows after an algebraic manipulation that

$$f(x) \leq M - 1/B$$

which is a contradiction to  $M$  being the **least** upper bound of the values of  $f$ . This completes the proof. The changes that should be made in the foregoing to prove that  $f$  assumes its minimum value are straightforward.

## Solved Problems

- 1) Show that the function  $f(x) = x^3 - x$  has absolute maximum and minimum values on the interval  $[-1, 1]$ .
- 2) Show that the function  $f(x) = 1/x$  assumes its absolute maximum and minimum values on the interval  $[1, 2]$  and find these extreme values.
- 3) Show that the function

$$f(x) = \begin{cases} x & 0 < x \leq 1 \\ 1 & -1 \leq x \leq 0 \end{cases}$$

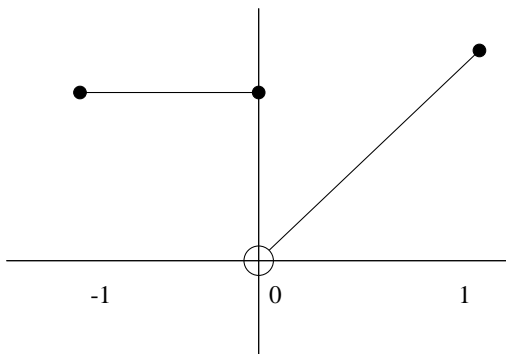
has no minimum and explain why the extreme value theorem doesn't apply.

## Answers:

- 1) The function  $f$  is a polynomial and is therefore continuous everywhere, including the given interval. The interval  $[-1, 1]$  is clearly bounded because every point in the interval satisfies  $-1 \leq x \leq 1$ , which makes  $-1$  a lower bound and  $+1$  an upper bound. Also, the interval is closed because the only boundary points,  $-1$  and  $1$ , belong to the interval. Therefore, by the extreme value theorem, the function  $f$  takes on its maximum and minimum values in the interval.
- 2) The given function is continuous on the interval  $[1, 2]$  because it is the reciprocal of a continuous function, i.e. the function given by  $x$ , which does not vanish anywhere on the interval. The interval  $[1, 2]$  is bounded below by the number 0 and above by the number 2 (note that a stated bound does not have to be the largest or smallest one). Furthermore the boundary points of the interval  $[1, 2]$  are 1 and 2, and they are in the interval.

Thus  $f$  is a continuous function on a closed bounded set and the extreme value theorem allows us to conclude that  $f$  has maximum and minimum values. Since  $f$  is strictly decreasing, neither extreme value can occur in the interior of the interval. To see this suppose that the maximum value occurred at  $c$  where  $1 < c < 2$ . Then if  $x$  were any number between 1 and  $c$ , we would have  $\frac{1}{x} > \frac{1}{c}$  which contradicts  $c$  being an absolute maximum. The extreme values can only occur at the endpoints. However  $f(1) > f(2)$ . Therefore  $f(1)$  must be the maximum and  $f(2)$  must be the minimum.

- 3) The graph of  $f$  looks like:



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We are going to examine some possibilities  $c$  for the minimum point. If  $-1 \leq c \leq 0$  then  $f(c) = 1$  and for  $x = \frac{1}{2}$  we have  $f(\frac{1}{2}) < f(c)$  so that such a  $c$  cannot be a point where the minimum is assumed.

If  $0 < c \leq 1$  then  $f(\frac{c}{2}) = \frac{c}{2} < f(c) = c$ . consequently such a point  $c$  cannot be where the minimum is assumed either. We have to conclude that there is no minimum because we have exhausted all the candidates for the minimum point.

Now recall the hypothesis of the extreme value theorem. Our function  $f$  is defined on  $[-1, 1]$  which is a closed bounded interval in keeping with the requirements of the theorem. However  $f$  is not continuous at the point 0 because

$$\lim_{x \rightarrow 0^-} f(x) = 1$$

but

$$\lim_{x \rightarrow 0^+} f(x) = 0$$

and these one sided limits are different.