4. An Inventory Model

In this section we shall construct a simple quantitative model to describe the cost of maintaining an inventory. Suppose you must meet an annual demand of V units of a certain product for which the rate of demand is constant throughout the year. Suppose further that you replenish your stock periodically throughout the year by ordering x units of the product just when your stock is about depleted. In this case a graph of your inventory level versus time would look something like Figure 1.
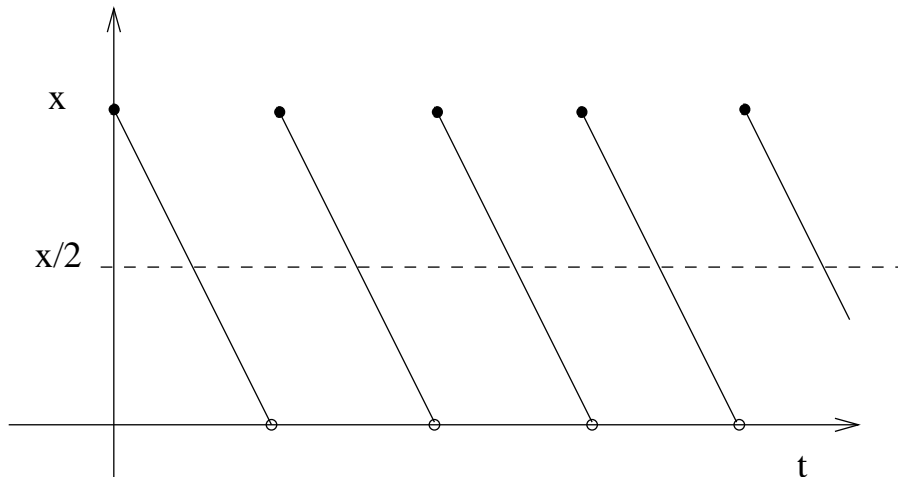

Figure 1

The decreasing lines are parallel since the rate of demand is constant throughout the year.

We shall incorporate three costs into our model: storage costs, re-ordering costs, and purchasing costs.

Storage Costs. Suppose the cost in dollars per annum of storing one unit is A. The average inventory level (see Figure 1) is $\frac{x}{2}$, so we shall take the annual storage cost to be

$$\frac{Ax}{2} \, .$$

Re-ordering Costs. Suppose the fixed cost in dollars of placing an order is B. If we order x units at a time, we must order $\frac{V}{x}$ times per year, so the re-ordering cost is

$$\frac{BV}{x} \, .$$

Purchasing Costs. Let $p(x)$ be the cost in dollars of purchasing x units. In practice this cost may involve discounts as an incentive to place large orders [see Example 2, Exercises 4, 5],

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but for this model let us assume that the per unit cost of purchasing x units is fixed— that is, $p(x) = kx$ — so that the total cost of purchasing V units is

$$\frac{V}{x} \cdot kx \;=\; Vk \;.$$

This is simply the cost of purchasing V units, and is a constant.

If $C(x)$ is the cost of maintaining the inventory, we then have

$$C(x) \;=\; \frac{Ax}{2} + \frac{BV}{x} + kV \;.$$

Mathematically we can use this model to *minimize* annual inventory costs.

$$
\begin{aligned}
C'(x) \;&=\; \frac{A}{2} - \frac{BV}{x^2} \\
&=\; 0 \\
\Leftrightarrow \qquad x^2 \;&=\; \frac{2BV}{A} \\
\Rightarrow \qquad x \;&=\; \sqrt{\frac{2BV}{A}}
\end{aligned}
$$

This value of x minimizes C since $C''(x) = \dfrac{2BV}{x^3}$ is positive (since we can assume x is positive), and is called the *economic lot size.* Note that the economic lot size is independent of purchasing cost, so that our simple model depends only on storage and re-ordering costs.

Example 1. A department store sells 500 refrigerators per year. The annual storage and carrying cost per unit is \$30 and the fixed reorder costs are \$50. At present, lots of 100 are ordered. How much can be saved by an adjustment of the order size?

Solution. Present annual costs are

$$C(100) \;=\; 1500 + \frac{50(500)}{100} \;=\; 1750 \;.$$

For a general lot size x, annual cost would be, in dollars,

$$C(x) \;=\; 15x + \frac{25,000}{x} \;.$$

A minimum is reached if

$$
\begin{aligned}
x \;&=\; \sqrt{\frac{2(500)(50)}{30}} \\
&=\; \sqrt{\frac{25,000}{15}} \\
&\doteq\; 40.8
\end{aligned}
$$

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Since x must be an integer, we calculate

$$\begin{array}{rclclcl} C(41) & = & 615 + \frac{25,000}{41} & = & 615 + 609.76 & = & \$1224.76 \\ C(40) & = & 600 + \frac{25,000}{40} & = & 600 + 625 & = & \$1225 \quad . \end{array}$$

Judgment suggests that the round number of 40 units is worth the 26 cents per year of savings foregone. Hence the saving by a change from 100 to 40 units per order is $1750 - 1225 = 625$ dollars per year.

In this way 25 orders are placed over each two-year period.

In this inventory problem we have assumed that the units do not deteriorate while in stock. Clearly such an assumption could not be made for perishable items. We have also supposed that an exact delivery time could be calculated to avoid overlap of stock, or shortage. While such assumptions are reasonable in simple inventory problems, it will often be found that further study in a real situation can lead to improved savings by taking these and other aspects into account in a more detailed model.

Example 2. A retailer expects to sell 1200 blenders per year. The wholesale price in dollars of purchasing a lot of x blenders is

$$p(x) = \begin{cases} 20x, & \text{if} & 0 \le x < 200 \\ 19x, & \text{if} & 200 \le x < 400 \\ 18x, & \text{if} & x \ge 400. \end{cases}$$

The storage cost for one blender is \$4 per year, and the ordering cost is \$50 per order. Since the per unit cost of purchasing x units is not constant, we must include the purchasing cost in total cost.

$$\begin{aligned} C(x) & = 2x + \frac{(50)(1200)}{x} + \frac{1200}{x} \cdot p(x) \\ & = 2x + \frac{60,000}{x} + \begin{cases} 24,000, & 0 < x < 200 \\ 22,800, & 200 \le x < 400 \\ 21,600, & x \ge 400. \end{cases} \end{aligned}$$
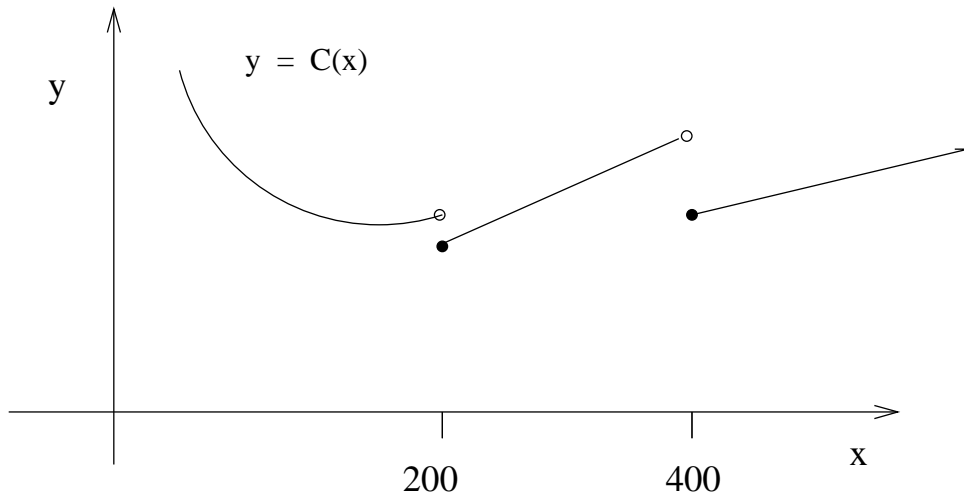
Since $C(x)$ is a discontinuous function, we must find the minimum of $C(x)$ in each of the intervals for which $C(x)$ is defined.

Now, $C'(x) = 2 - \dfrac{60,000}{x^2}$, for $x \ne 0$, 200, or 400.

$$C'(x) = 0 \quad \Rightarrow \quad x = \sqrt{30,000}$$

$$\doteq 173 .$$

(Economic lot size for $A = 4$, $B = 50$, $V = 1200$.)

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Hence the minimum of C is at $x = 173, 200$ or 400.

We compare the values to find

$$C(173) = 346 + 346.82 + 24,000$$

$$= 24,692.82$$

$$C(200) = 400 + 300 + 22,800$$

$$= 23,500$$

$$\text{and} \quad C(400) = 800 + 150 + 21,600$$

$$= 22,550 .$$

So the minimum is attained for $x = 400$, which is the lot size that takes full advantage of the wholesaler's discount.

Exercises

1. An agency sells 10,000 brushes per year. Storage costs are \$50 per thousand per year and reorder cost is \$16 per order. What lot size should be ordered?

2. A company's storage costs increase by 10% and its reorder costs by 6%. Also sales will increase by 8%. Assuming optimal lot size at present, by what percentage should the next year's lot size be increased.

3. A rancher is offered \$1.00 per kg. for his flock of 1000 sheep. They weigh, on the average, 100 kg. and are increasing in weight at the rate of $\frac{1}{2}$ kg. per day. The cost of maintaining one sheep for a day is 50¢ and the market price is expected to fall by 2¢ per kg. per day. How long should the rancher postpone the sale of his sheep? How much can he gain in comparison with his present offer?

4. Recalculate the optimal lot size in Example 2 if the number of blenders sold per year changes to

<div align="center">(a)   4, 800   (b)   12, 000 .</div>

5. A building supply store expects to sell 12,000 bags of cement this year. Storage cost on the premises is \$5 per bag per year and reorder costs are \$100. The manufacturer offers a discount for volume of each order as follows:

Price, delivered: (in dollars)

\$8.00	per bag for up to	99	bags	
7.75	per bag from	100	to	199 bags
7.50	per bag from	200	to	499 bags
7.25	per bag from	500	and	above.

Find the optimal lot size.