

3. Profit Under a Monopoly

Under a monopoly a producer has full control over how many units of a product will reach market. Since market demand and selling price are related by a demand relation, the monopolist can also control the price at which the commodity sells. Under a free enterprise system it is assumed that the monopolist will attempt to maximize his or her profit. How many units should the monopolist produce?

Let $R(x)$ be the revenue associated with the sale of x units; $C(x)$ the cost to the monopolist of producing x units. Then profit is defined to be $P(x)$, with $P(x) = R(x) - C(x)$. In Figure 1 typical revenue and cost curves are shown, and the profit zone is shaded in.

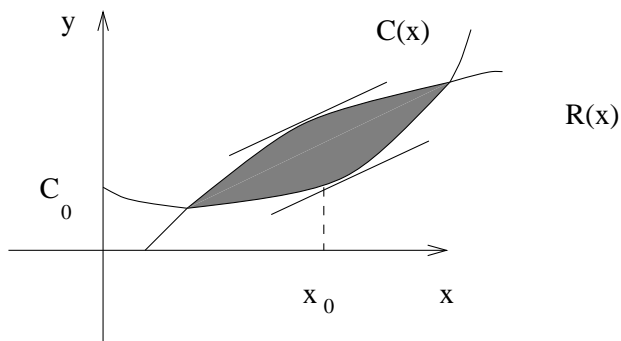


Figure 1

Now $P'(x) = R'(x) - C'(x)$, and $P'(x) = 0$ if and only if $R'(x) = C'(x)$. So as long as $P''(x) < 0$, the profit will be maximized at the production level for which marginal revenue equals marginal cost. In Figure 1 this optimal production level is denoted by x_0 ; it is the point where the tangents to the revenue and cost curves are parallel.

Note that the values of price and volume for optimal profit do not depend on the overhead cost, but if the overhead cost is too high no positive profit may be possible for any realistic price.

This can be seen in Figure 1 where we have called the overhead cost C_0 . Increasing C_0 may eventually place the entire cost curve above the revenue curve so that there would be no profitable region.

Example 1. A publisher estimates the cost of producing and selling x copies of a new science fiction book will be, in dollars,

$$C(x) = 10,000 + 5x .$$

The demand of the book at price p dollars is

$$x = 4,000 - 200p .$$

Find the most profitable sale price.

Solution. Solving the demand relation for p , we find

$$p = 20 - \frac{x}{200} \quad \text{dollars .}$$

Hence total revenue is

$$R(x) = 20x - \frac{x^2}{200}$$

and profit is

$$P(x) = 20x - \frac{x^2}{200} - 5x - 10,000 .$$

For maximum value,

$$\begin{aligned} P'(x) &= 15 - \frac{x}{100} = 0 \\ \therefore x &= 1500 . \end{aligned}$$

Observe that

$$P''(x) = -\frac{1}{100} < 0 \quad \text{for all } x ,$$

so that a true maximum value has been found. The price is then $p = 20 - \frac{1500}{200} = 12.50$ (dollars) and the maximum profit is

$$\begin{aligned} P(1500) &= 15(1500) - \frac{(1500)^2}{200} - 10,000 \\ &= 1,250 \quad \text{(dollars) .} \end{aligned}$$

Example 2. A Case Study of Taxation on a Monopoly

In this example we shall construct a simple model to analyse taxation on a monopoly. We assume that the cost and demand curves for a certain product are both linear, and that a tax of t dollars per unit is levied on the producer for every unit produced.

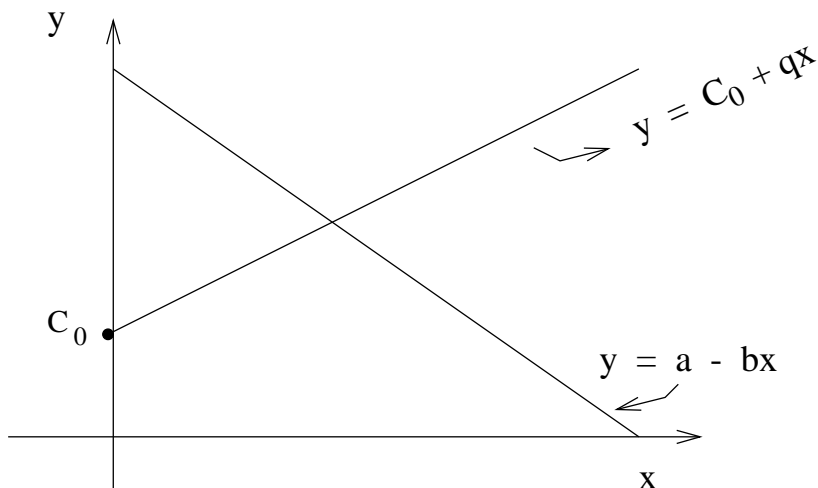


Figure 2

Let the demand curve be $p = a - bx$, with $a > 0$, $b > 0$; and let the cost curve be $C = C_0 + qx$, where C_0 denotes the overhead cost and q is positive. With tax t dollars per unit, the producer's cost function becomes

$$C = C_0 + qx + tx .$$

The revenue function is xp , so the profit function is

$$\begin{aligned} P(x) &= xp - C \\ &= x(a - bx) - C_0 - (q + t)x \\ &= -C_0 + (a - q - t)x - bx^2 . \\ P'(x) &= a - q - t - 2bx \\ &= 0 \\ \Leftrightarrow x &= \frac{a - q - t}{2b} . \end{aligned}$$

Since $P''(x) = -2b$, which is negative, the monopolist's profit is maximized by producing $\frac{a - q - t}{2b}$ units. The maximum profit is

$$P\left(\frac{a - q - t}{2b}\right) = \frac{(a - q - t)^2}{4b} - C_0 .$$

This profit is positive provided that

$$C_0 \leq \frac{(a - q - t)^2}{4b} ,$$

which for $t = 0$ becomes

$$C_0 \leq \frac{(a - q)^2}{4b} .$$

Suppose the government sets the tax t to maximize its revenue, which is

$$T(t) = tx = t \frac{(a - t - q)}{2b} = \frac{(a - q)}{2b}t - \frac{t^2}{2b} .$$

Then t is chosen so that

$$\begin{aligned} T'(t) &= \frac{a - q}{2b} - \frac{t}{b} = 0 \\ \Leftrightarrow t &= \frac{a - q}{2} . \end{aligned}$$

Then

$$T(t) = \frac{(a - q)^2}{8b} \quad \text{and} \quad P_{\text{Max}} = \frac{(a - q)^2}{16b} - C_0 = \frac{1}{2}T(t) - C_0 .$$

To summarize:

1. The producer's volume for maximum profit is

$$x = \frac{a - t - q}{2b} .$$

2. The tax for maximum tax revenue is $t = \frac{a - q}{2}$. Then $x = \frac{a - q}{4b}$ becomes half of its value in the absence of taxation.
3. Since $P_{\text{Max}} = \frac{1}{2}T(t) - C_0$, the profit is less than the tax revenue.
4. In the absence of taxation ($t = 0$), the greatest overhead cost compatible with any profit is

$$C_0 \leq \frac{(a - q)^2}{4b} .$$

If $t = \frac{a - q}{2}$, the greatest overhead cost compatible with any profit is

$$C_0 \leq \frac{(a - q)^2}{16b}$$

which is one quarter of the previous amount.

Observe that the simple algebra of these relationships obtained by optimization in x and in t permits detailed qualitative analysis of the possible extreme cases. However such results should be interpreted with caution because the linear demand and cost assumptions might not apply accurately in any given situation.

Exercises

- For a charter holiday trip by bus, a travel agent charts a bus with capacity 60 seats for \$2000. He sells x tickets at \$100 each with the understanding that \$ x is refunded to each passenger when the trip begins. Find
 - The minimum number of tickets he must sell to break even,
 - the most profitable number of tickets he can sell.
- The demand relation for handsaws is that x saws sell per year at the price $p = 8 - 0.0004x$ dollars. The total cost of producing x saws and offering them for sale is $1200 + 6x$ dollars. If annual profit is to be maximized, determine the number x of units produced per year, the price and the total profit.
- The monthly demand for soy beans is x million bushels at a price $p = (x - 6)^2$ dollars per bushel, where $0 \leq x \leq 6$. The cost of growing and storing x million bushels is $C(x) = 12x - x^2$ million dollars.
 - What size of crop will yield the greatest profit? What profit?
 - What range of crop sizes will yield a profit?
- A book publisher estimates that a paperback almanac will sell x thousand copies if priced at $p = 5 - \frac{x}{10}$ dollars. The total publishing and distributing costs are, in thousands of dollars, $C(x) = 125 - 2x$. Find the recommended price.
- The demand equation for frisbees is $p = 2 - 0.0001x$ dollars, when x units are sold per week. The total cost of producing x units is estimated at $300 + 1.5x$. For maximum weekly profit, find
 - the number x of units to be produced per week
 - the unit price, and
 - the weekly profit.
- The government imposes a tax of 10 cents on each frisbee produced in Exercise 5. Find
 - the new production volume x for best profit
 - the new price
 - the ratio of the price increase from the level of Exercise 5 to the tax imposed
 - the total tax collected
 - the weekly profit.
- After a change of tax law the tax on frisbees in Exercise 5 becomes a 6% sales tax. Repeat Exercise 5 for this new tax.

8. How much tax t should the government impose on each frisbee to maximize its tax revenue? Determine the producer's optimum production level x , total profit, and unit price if the government chooses to maximize tax revenue.
9. Repeat the analysis of the case study in Example 2 when $C(x) = C_0 + qx$, the demand curve is $p = a - bx^m$, m positive, and unit tax is t (dollars). Show that maximum tax revenue T_m is given by $T_m = (m + 1)(P_{\text{Max}} + C_0)$ where P_{Max} is the best profit for the producer at the given tax rate.
10. Repeat the analysis of the case study in Example 2 when $C(x) = C_0 + qx^{m+1}$ and the demand curve is $p = a - bx^m$, m positive, as in Exercise 9. Show that T_m and P_{Max} satisfy the same relation as in Exercise 9.