FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL 2016

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

Duration: 3 hours
Examiners: N. Hoell, A. Igelfeld, D. Reiss, L. Shorser, J. Tate

LEAVE BLANK	
Question	Mark
MC	/45
B1	/10
B2	/14
B3	/10
B4	/10
B5	/11
BONUS	/5
TOTAL	

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

NOTE:

1. **Aids Allowed:** A non-graphing calculator, with empty memory, to be supplied by student. No calculator may be used that has a button with $\frac{d}{dx}$ and/or $\int$ on it.
2. **Instructions:** Fill in the information on this page, and make sure your test booklet contains 14 pages.
3. This exam consists of 15 multiple choice questions, and 5 written-answer questions. For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the front page** with your pencil. Each correct answer is worth 3 marks; a question left blank, or an incorrect answer or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.
4. Put your name and student number on each page of this examination.

ANSWER BOX FOR PART A

Circle the correct answer

1.	**A.**	**B.**	**C.**	**D.**	**E.**
2.	**A.**	**B.**	**C.**	**D.**	**E.**
3.	**A.**	**B.**	**C.**	**D.**	**E.**
4.	**A.**	**B.**	**C.**	**D.**	**E.**
5.	**A.**	**B.**	**C.**	**D.**	**E.**
6.	**A.**	**B.**	**C.**	**D.**	**E.**
7.	**A.**	**B.**	**C.**	**D.**	**E.**
8.	**A.**	**B.**	**C.**	**D.**	**E.**
9.	**A.**	**B.**	**C.**	**D.**	**E.**
10.	**A.**	**B.**	**C.**	**D.**	**E.**
11.	**A.**	**B.**	**C.**	**D.**	**E.**
12.	**A.**	**B.**	**C.**	**D.**	**E.**
13.	**A.**	**B.**	**C.**	**D.**	**E.**
14.	**A.**	**B.**	**C.**	**D.**	**E.**
15.	**A.**	**B.**	**C.**	**D.**	**E.**

Name: ______________________________ Student #: ____________________

Record your answers on the front page

PART A. MULTIPLE CHOICE

1. *[3 marks]*
If the demand function is given by $p = \dfrac{500}{\sqrt{q}}$ and the total cost function by $c = 5q + 2000$ then profit is maximized when $q =$

   A. 50

   B. 100

   C. 750

   D. 1235

   E. 2500

2. *[3 marks]*
Find the equation(s) of the tangent line(s) to the curve $f(x) = x^3 - 3x$ that are parallel to the x-axis.

   A. $x = 0, x = -\sqrt{3}, x = \sqrt{3}$

   B. $y = -2, y = 2$

   C. $y = -2$ only

   D. $x = -1, x = 1$

   E. $y = -1, y = 1$

3. *[3 marks]*

   Given $\dfrac{y^2}{x} = y \cdot 2^{\sqrt{x}} + 4x - 4$ then at $x = 1$ and $y = 2$, $\dfrac{dy}{dx} =$

   A. $\ln 2 + 4$

   B. $\ln 2 + 2$

   C. 5

   D. $\ln 2 + 8$

   E. $-2 - 2\ln 2$

4. *[3 marks]*

   The curve $f(x) = \dfrac{x}{e^x}$ is concave up when

   A. $x > -1$ only

   B. $x < 2$ only

   C. $x < 1$ only

   D. $x > -2$ only

   E. $x > 2$ only

Record your answers on the front page

5. *[3 marks]*

If $a > 0$ is a constant, $\lim_{x \to \infty} \frac{a^{\frac{2}{x}} - 1}{a^{\frac{3}{x}} - 1} =$

A. does not exist

B. $\frac{1}{5}$

C. $\frac{2}{3}$

D. $\frac{1}{3}$

E. 2

6. *[3 marks]*

Let $G(x) = \int_1^x \ln t \, dt$ for $1 < x < 10$. Then $G(x)$ has:

A. no relative extrema nor absolute extrema

B. an absolute maximum, but no absolute minimum nor relative extrema

C. an absolute minimum, but no absolute maximum nor relative extrema

D. a relative maximum, but no absolute extrema

E. a relative minimum, but no absolute extrema

7. *[3 marks]*
The integral $\int_0^1 (6x+1)\, e^{3x^2+x-4} dx$ is equal to

A. $\int_0^4 e^w dw$

B. $\int_{-4}^0 e^w dw$

C. $\int_0^1 e^w dw$

D. $\int_1^7 e^w dw$

E. $\int_1^{-4} e^w dw$

8. *[3 marks]*
Find the average value of $g(s) = \sqrt{s}$ on the interval $[0, 9]$.

A. $\frac{3}{2}$

B. $\sqrt{\frac{9}{2}}$

C. 6

D. 2

E. 3

9. *[3 marks]*
Find the present value of a continuous annuity at an annual rate of 3% compounded continuously for four years if the payment at time t is at the rate of \$400 per year.

A. \$1384.07

B. \$1679.34

C. \$1592.20

D. \$1507.73

E. \$1608.44

10. *[3 marks]*
$\int_4^\infty \frac{1}{x^{\frac{3}{2}}} dx =$

A. 1

B. $\frac{1}{8}$

C. $\frac{3}{8}$

D. 0

E. diverges

Record your answers on the front page

11. *[3 marks]*
If $f(x, y, z, w) = 17xyzw + 17yzw - 17yw$, then $f_x =$

A. $17yzw$

B. 17

C. $17yzw + 17x$

D. 0

E. $17x$

12. *[3 marks]*
Assume that $x, y, z > 0$ and x, y, and z are independent variables. Then $\frac{\partial}{\partial r}\left(x^2 y^r z^6\right) =$

A. 0

B. $r x^2 y^{r-1} z^6$

C. $x^2 y^r z^6 \ln y$

D. $x^2 y^r z^6 \ln r$

E. $\ln y$

13. *[3 marks]*
The joint demand functions for products A and B are

$q_A = \dfrac{10\sqrt{p_B}}{p_A}$

$q_B = 20 + 3p_A - 2p_B$

respectively.

A. A and B are neither competitive nor complementary when $20 + 3p_A - 2p_B = 10$

B. A and B are competitive as long as $20 + 3p_A - 2p_B > 10$ and complementary as long as $20 + 3p_A - 2p_B < 10$

C. A and B are neither competitive nor complementary at any positive prices.

D. A and B are competitive at all positive prices.

E. A and B are complementary at all positive prices.

14. *[3 marks]*
If x, y, and z are independent variables $\dfrac{\partial^2}{\partial x \partial y}(xyz)^6 =$

A. 0

B. $6(xz)^6 y^5$

C. $30x^5y^5z^6$

D. $6(xy)^5 z^6$

E. $36x^5y^5z^6$

15. *[3 marks]*
Let $z = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$.
It is given that

$$g(1,0) = 3 \qquad g_r(1,0) = \frac{1}{2} \qquad g_s(1,0) = \frac{1}{3}$$

$$g(1,1) = 2 \qquad g_r(1,1) = \frac{1}{5} \qquad g_s(1,1) = -1$$

$$h(1,0) = 1 \qquad h_r(1,0) = \frac{1}{4} \qquad h_s(1,0) = \frac{1}{5}$$

$$h(1,1) = 0 \qquad h_r(1,1) = 1 \qquad h_s(1,1) = -2$$

$$f(1,0) = 6 \qquad f_x(1,0) = \frac{1}{6} \qquad f_y(1,0) = \frac{1}{7}$$

$$f(1,1) = -1 \qquad f_x(1,1) = -\frac{2}{3} \qquad f_y(1,1) = \frac{1}{4}$$

$$f(3,1) = 5 \qquad f_x(3,1) = 3 \qquad f_y(3,1) = -2$$

At $r = 1$ and $s = 0$, $\dfrac{\partial z}{\partial r} =$

A. $-\dfrac{4}{3}$

B. $-\dfrac{3}{10}$

C. $\dfrac{5}{21}$

D. $\dfrac{6}{5}$

E. 1

Name: ____________________ Student #: ____________________

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[10 marks]*

An entrepreneur has an account that guarantees him 3% per year compounded annually on any money he places in it for the next 15 years. 10 years from now, he will need to withdraw $\$300,000$, and 15 years from new, another $\$100,000$. He is due to receive $\$20,000$ at the end of each year for the first 10 years and will place the proceeds into the account. To meet the remaining part of the required payments, he will sell used equipment after the first 3 years and also place the proceeds into the account. How much money must he realize from selling the equipment to meet his obligations?

Name: ______________________ Student #: ______________

B2. *[14 marks]*

(a) *[7 marks]*
The production function for a company is given by

$$P(\ell, k) = 20\ell^{\frac{1}{3}}k^{\frac{2}{3}}$$

where ℓ denotes units of labour and k denotes units of capital.
Labour costs \$225/unit and capital costs \$150/unit and the total cost of labour and capital is \$270,000. Use Lagrange multipliers to find the number of units of labour and capital that maximize production.
You don't have to show that your answer is indeed a maximum.

(b) *[7 marks]*
Find and classify the critical points of

$$f(x, y) = xy + \frac{8}{y} + \frac{8}{x} \quad (x \neq 0, y \neq 0)$$

B3. *[10 marks]*

Find the area of the finite region bounded by the graphs of $x = y^2 + 3y - 4$ and $y = 1 - x$.

Name: ______________________________ Student #: ______________

B4. *[10 marks]*

(a) *[7 marks]*
If $y = f(x)$ is the most general solution to the differential equation

$$\frac{dy}{dx} = \frac{3y}{x} \quad \text{for } x, y > 0$$

find $f(x)$.

(b) *[3 marks]*
What is the answer if $f(1) = 2$?

Name: ______________________________ Student #: ______________

B5. *[11 marks]*

Evaluate

(a) *[5 marks]*

$$\int x^2 \ln x dx$$

(b) *[6 marks]*

$$\int \frac{3x+1}{x(x+1)^2} dx$$

Bonus: *[extra 5 marks possible]*

$$\int_1^{\infty} \frac{3x+1}{x(x+1)^2} dx$$

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Circle the correct answer					
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Name: ____________________ Student #: ____________________

PART A. MULTIPLE CHOICE

B1. *[3 marks]*

If the demand function is given by $p = \dfrac{500}{\sqrt{q}}$ and the total cost function by $c = 5q + 2000$ then profit is maximized when $q =$

A. 50

B. 100

C. 750

D. 1235

E. 2500

$$\begin{aligned}
\pi &= R - C \\
&= pq - c \\
&= \frac{500}{\sqrt{q}}q - (5q + 2000) \\
&= 500\sqrt{q} - 5q - 2000 \\
\frac{d\pi}{dq} &= \frac{250}{\sqrt{q}} - 5 = 0 \text{ when } \sqrt{q} = 50 \text{ or } q = 2500
\end{aligned}$$

Note that on $(0, 2500)$ $\dfrac{d\pi}{dq} > 0$ and on $(2500, \infty)$ $\dfrac{d\pi}{dq} < 0$. So π increases from 0 to 2500 and decreases forever afterwards. Maximum is at $q = 2500$, **E**

B2. *[3 marks]*

Find the equation(s) of the tangent line(s) to the curve $f(x) = x^3 - 3x$ that are parallel to the x-axis.

A. $x = 0, x = -\sqrt{3}, x = \sqrt{3}$

B. $y = -2, y = 2$

C. $y = -2$ only

D. $x = -1, x = 1$

E. $y = -1, y = 1$

Parallel to x-axis means slope $= 0$, and the equation(s) are $y = c$ constant. When is slope $= 0$? When $f'(x) = 0$, $3x^2 - 3 = 0$, $x^2 - 1 = 0$, $x = -1$, or $x = 1$.
At $x = -1$, $y = 2$, line is $y = 2$.
At $x = 1$, $y = -2$, line is $y = -2$.
B

Name: ______________________________ Student #: ______________

B3. *[3 marks]*

Given $\dfrac{y^2}{x} = y \cdot 2^{\sqrt{x}} + 4x - 4$ then at $x = 1$ and $y = 2$, $\dfrac{dy}{dx} =$

A. $\ln 2 + 4$

B. $\ln 2 + 2$

C. 5

D. $\ln 2 + 8$

E. $-2 - 2\ln 2$

$$\frac{x \cdot 2yy' - y^2}{x^2} = y' \cdot 2^{\sqrt{x}} + y \cdot 2^{\sqrt{x}} \ln(2) \frac{1}{2\sqrt{x}} + 4$$

$x = 1$, $y = 2$ gives

$$\begin{aligned} 4y' - 4 &= y' \cdot 2 + \frac{4\ln(2)}{2} + 4 \\ 2y' &= 2\ln(2) + 8 \\ y' &= \ln(2) + 4 \quad \textbf{A} \end{aligned}$$

B4. *[3 marks]*

The curve $f(x) = \dfrac{x}{e^x}$ is concave up when

A. $x > -1$ only

B. $x < 2$ only

C. $x < 1$ only

D. $x > -2$ only

E. $x > 2$ only

$$\begin{aligned} f''(x) &> 0 \quad f(x) = xe^{-x} \\ f'(x) &= e^{-x} - xe^{-x} \\ f''(x) &= -e^{-x} - e^{-x} + xe^{-x} \\ &= e^{-x}(x-2) > 0 \text{ when } x > 2 \text{ only } \textbf{E} \end{aligned}$$

Name: ______________________________ Student #: ______________

B5. *[3 marks]*

If $a > 0$ is a constant, $\lim_{x\to\infty} \dfrac{a^{\frac{2}{x}} - 1}{a^{\frac{3}{x}} - 1} =$

- **A.** does not exist
- **B.** $\dfrac{1}{5}$
- **C.** $\dfrac{2}{3}$
- **D.** $\dfrac{1}{3}$
- **E.** 2

$$\lim_{x\to\infty} a^{\frac{2}{x}} = a^0 = 1$$
$$\lim_{x\to\infty} a^{\frac{3}{x}} = a^0 = 1$$

So $\dfrac{0}{0}$, use L'Hôpital.

$$= \lim_{x\to\infty} \frac{a^{\frac{2}{x}} \ln(a) \left(\frac{-2}{x^2}\right)}{a^{\frac{3}{x}} \ln(a) \left(\frac{-3}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{2}{3} \frac{a^{\frac{2}{x}}}{a^{\frac{3}{x}}} = \frac{2}{3}\frac{a^0}{a^0} = \frac{2}{3} \qquad \mathbf{C}$$

B6. *[3 marks]*

Let $G(x) = \displaystyle\int_1^x \ln t \, dt$ for $1 < x < 10$. Then $G(x)$ has:

- **A.** no relative extrema nor absolute extrema
- **B.** an absolute maximum, but no absolute minimum nor relative extrema
- **C.** an absolute minimum, but no absolute maximum nor relative extrema
- **D.** a relative maximum, but no absolute extrema
- **E.** a relative minimum, but no absolute extrema

By the Fundamental Theorem of Calculus
$G'(x) = \ln x > 0$ on $(1, 10)$
G is an increasing function and has no extrema of any kind on the open interval **A**

Name: ________________________ Student #: ________________

B7. *[3 marks]*

The integral $\int_0^1 (6x+1)\, e^{3x^2+x-4} dx$ is equal to

A. $\int_0^4 e^w dw$

B. $\int_{-4}^0 e^w dw$

C. $\int_0^1 e^w dw$

D. $\int_1^7 e^w dw$

E. $\int_1^{-4} e^w dw$

Let $w = 3x^2 + x - 4$, $dw = (6x+1)dx$
$x = 0 \implies w = -4, x = 1 \implies w = 0$

$$\int_0^1 (6x+1)e^{3x^2+x-4}dx = \int_{-4}^0 e^w dw \quad \mathbf{B}$$

B8. *[3 marks]*

Find the average value of $g(s) = \sqrt{s}$ on the interval $[0, 9]$.

A. $\frac{3}{2}$

B. $\sqrt{\frac{9}{2}}$

C. 6

D. 2

E. 3

$$\begin{aligned} \text{Average} &= \frac{1}{9}\int_0^9 \sqrt{s} ds \\ &= \frac{1}{9}\frac{2}{3}s^{3/2}\Big|_0^9 \\ &= \frac{2}{27}9^{3/2} \\ &= 2 \qquad \mathbf{D} \end{aligned}$$

Name: ____________________ Student #: ____________________

B9. *[3 marks]*
Find the present value of a continuous annuity at an annual rate of 3% compounded continuously for four years if the payment at time t is at the rate of \$400 per year.

A. \$1384.07

B. \$1679.34

C. \$1592.20

D. \$1507.73

E. \$1608.44

$$\begin{aligned}
\text{P.V.} &= \int_0^T f(t)e^{-rt}dt \\
&= \int_0^4 400e^{-0.03t}dt \\
&= \frac{400}{-0.03}e^{-0.03t}\Big|_0^4 \\
&= \frac{400}{-0.03}\left(e^{-0.12}-1\right) \\
&= \$1507.73 \qquad \textbf{D}
\end{aligned}$$

B10. *[3 marks]*
$\displaystyle\int_4^\infty \frac{1}{x^{\frac{3}{2}}}dx =$

A. 1

B. $\frac{1}{8}$

C. $\frac{3}{8}$

D. 0

E. diverges

$$\begin{aligned}
\lim_{R\to\infty}\int_4^R x^{-3/2}dx &= \lim_{R\to\infty} -2x^{-1/2}\Big|_4^R \\
&= \lim_{R\to\infty}\left(\frac{-2}{\sqrt{R}}+\frac{2}{\sqrt{4}}\right) = 1 \qquad \textbf{A}
\end{aligned}$$

Name: ______________________________ Student #: ______________

B11. *[3 marks]*

If $f(x, y, z, w) = 17xyzw + 17yzw - 17yw$, then $f_x =$

A. $17yzw$

B. 17

C. $17yzw + 17x$

D. 0

E. $17x$

Only the first term depends on x:
$f_x = 17yzw$ **A**

B12. *[3 marks]*

Assume that $x, y, z > 0$ and x, y, and z are independent variables. Then $\frac{\partial}{\partial r}\left(x^2 y^r z^6\right) =$

A. 0

B. $rx^2 y^{r-1} z^6$

C. $x^2 y^r z^6 \ln y$

D. $x^2 y^r z^6 \ln r$

E. $\ln y$

$$\begin{aligned} \frac{\partial}{\partial r}\left(x^2 y^r z^6\right) &= x^2 z^6 \frac{\partial}{\partial r}\left(y^r\right) \\ &= x^2 z^6 y^r \ln y \quad \mathbf{C} \end{aligned}$$

Name: ______________________ Student #: ______________

B13. *[3 marks]*

The joint demand functions for products A and B are

$q_A = \dfrac{10\sqrt{p_B}}{p_A}$

$q_B = 20 + 3p_A - 2p_B$

respectively.

- **A**. A and B are neither competitive nor complementary when $20 + 3p_A - 2p_B = 10$
- **B**. A and B are competitive as long as $20 + 3p_A - 2p_B > 10$ and complementary as long as $20 + 3p_A - 2p_B < 10$
- **C**. A and B are neither competitive nor complementary at any positive prices.
- **D**. A and B are competitive at all positive prices.
- **E**. A and B are complementary at all positive prices.

$\dfrac{\partial q_A}{\partial p_B} = \dfrac{5}{p_A\sqrt{p_B}} > 0$ at all positive prices

$\dfrac{\partial q_B}{\partial p_A} = 3 > 0$ at all positive prices

The goods are competitive at all positive prices **D**

B14. *[3 marks]*

If x, y, and z are independent variables $\dfrac{\partial^2}{\partial x \partial y}(xyz)^6 =$

- **A**. 0
- **B**. $6(xz)^6y^5$
- **C**. $30x^5y^5z^6$
- **D**. $6(xy)^5z^6$
- **E**. $36x^5y^5z^6$

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y}(xyz)^6 &= \frac{\partial^2}{\partial x \partial y}\left(x^6y^6z^6\right) \\
&= \frac{\partial}{\partial x}\left(6x^6y^5z^6\right) \\
&= 36x^5y^5z^6 \quad \textbf{E}
\end{aligned}$$

Alternatively

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y}(xyz)^6 &= \frac{\partial}{\partial x}\left[6(xyz)^5xz\right] \\
&= 30(xyz)^4 \cdot yz \cdot xz + 6(xyz)^5z \\
&= 30x^5y^5z^6 + 6x^5y^5z^6 \\
&= 36x^5y^5z^6 \text{ as before}
\end{aligned}$$

Name: ______________________________ Student #: ______________________

B15. *[3 marks]*

Let $z = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$.
It is given that

$$\begin{array}{lll} g(1,0) = 3 & g_r(1,0) = \frac{1}{2} & g_s(1,0) = \frac{1}{3} \\ g(1,1) = 2 & g_r(1,1) = \frac{1}{5} & g_s(1,1) = -1 \end{array}$$

$$\begin{array}{lll} h(1,0) = 1 & h_r(1,0) = \frac{1}{4} & h_s(1,0) = \frac{1}{5} \\ h(1,1) = 0 & h_r(1,1) = 1 & h_s(1,1) = -2 \end{array}$$

$$\begin{array}{lll} f(1,0) = 6 & f_x(1,0) = \frac{1}{6} & f_y(1,0) = \frac{1}{7} \\ f(1,1) = -1 & f_x(1,1) = -\frac{2}{3} & f_y(1,1) = \frac{1}{4} \\ f(3,1) = 5 & f_x(3,1) = 3 & f_y(3,1) = -2 \end{array}$$

At $r = 1$ and $s = 0$, $\dfrac{\partial z}{\partial r} =$

A. $-\dfrac{4}{3}$
B. $-\dfrac{3}{10}$
C. $\dfrac{5}{21}$
D. $\dfrac{6}{5}$
E. 1

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= f_x g_r + f_y h_r \end{aligned}$$

We want $g_r(1,0) = \dfrac{1}{2}$ and $h_r(1,0) = \dfrac{1}{4}$
Also, when $r = 1$ and $s = 0$, $x = g(1,0) = 3$, $y = h(1,0) = 1$
So we want $f_x(3,1) = 3$ and $f_y(3,1) = -2$
$\dfrac{\partial z}{\partial r} = 3\left(\dfrac{1}{2}\right) - 2\left(\dfrac{1}{4}\right) = 1$ **E**

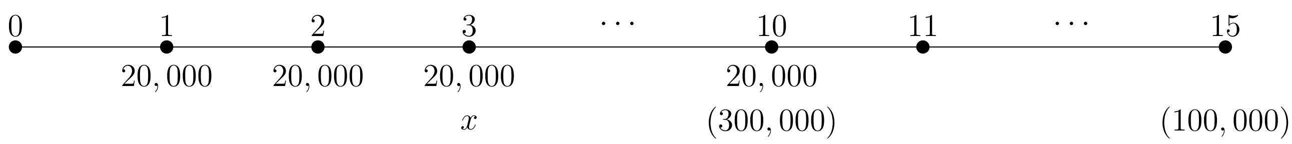
Name: ______________________ Student #: ______________

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[10 marks]*

An entrepreneur has an account that guarantees him 3% per year compounded annually on any money he places in it for the next 15 years. 10 years from now, he will need to withdraw \$300,000, and 15 years from new, another \$100,000. He is due to receive \$20,000 at the end of each year for the first 10 years and will place the proceeds into the account. To meet the remaining part of the required payments, he will sell used equipment after the first 3 years and also place the proceeds into the account. How much money must he realize from selling the equipment to meet his obligations?

Let x be the proceeds he will need from the sale of equipment.



Set Income = Expenses at any moment:
At 15, for example,

$$20,000 s_{\overline{10}|0.03}(1.03)^5 + x(1.03)^{12} = 300,000(1.03)^5 + 100,000$$

Solving for x,

$$\text{The equation at 3:} \qquad 20,000 s_{\overline{10}|0.03}(1.03)^{-7} + x = 300,000(1.03)^{-7} + 100,000(1.03)^{-12}$$

$$x = \$127,641.78$$

Alternatively,

$x(1.03)^7 + 20,000 s_{\overline{10}|0.03}$ is how much money he will have after 10 years. If he then pays 300,000, $\left[x(1.03)^7 + 20,000 s_{\overline{10}|0.03}\right] - 300,000$ remains in the account.
After a further 5 years,
$\left[x(1.03)^7 + 20,000 s_{\overline{10}|0.03} - 300,000\right](1.03)^5$ must equal 100,000.
Multiplying this out and solving for x gives the same answer (and is in fact just setting the equation of value at 15 instead of at 3.

Name: ______________________________ Student #: ____________________

B2. *[14 marks]*

(a) *[7 marks]*

The production function for a company is given by

$$P(\ell, k) = 20\ell^{\frac{1}{3}}k^{\frac{2}{3}}$$

where ℓ denotes units of labour and k denotes units of capital.
Labour costs \$225/unit and capital costs \$150/unit and the total cost of labour and capital is \$270,000. Use Lagrange multipliers to find the number of units of labour and capital that maximize production.
You don't have to show that your answer is indeed a maximum.

Budget constraint: $225\ell + 150k = 270,000$

$$\begin{aligned}
\mathcal{L} &= 20\ell^{\frac{1}{3}}k^{\frac{2}{3}} - \lambda(225\ell - 150k - 270,000) \\
\mathcal{L}_\ell &= \frac{20}{3}\ell^{\frac{-2}{3}}k^{\frac{2}{3}} - 225\lambda = 0 \quad \text{and } \mathcal{L}_\lambda = 0 \text{ is the budget constraint} \\
\mathcal{L}_k &= \frac{40}{3}\ell^{\frac{1}{3}}k^{\frac{-1}{3}} - 150\lambda = 0 \\
\frac{20}{3}\ell^{\frac{-2}{3}}k^{\frac{2}{3}} &= 225\lambda \\
\frac{40}{3}\ell^{\frac{1}{3}}k^{\frac{-1}{3}} &= 150\lambda \quad \text{dividing the second equation by the first} \\
2\frac{\ell}{k} &= \frac{150}{225} \\
\frac{\ell}{k} &= \frac{1}{3} \quad \text{so } k = 3\ell
\end{aligned}$$

Subbing into budget:

$$\begin{aligned}
225\ell + 450\ell &= 270,000 \\
675\ell &= 270,000 \\
\ell &= 400 \\
k &= 1200
\end{aligned}$$

(b) *[7 marks]*

Find and classify the critical points of

$$f(x, y) = xy + \frac{8}{y} + \frac{8}{x} \quad (x \neq 0, y \neq 0)$$

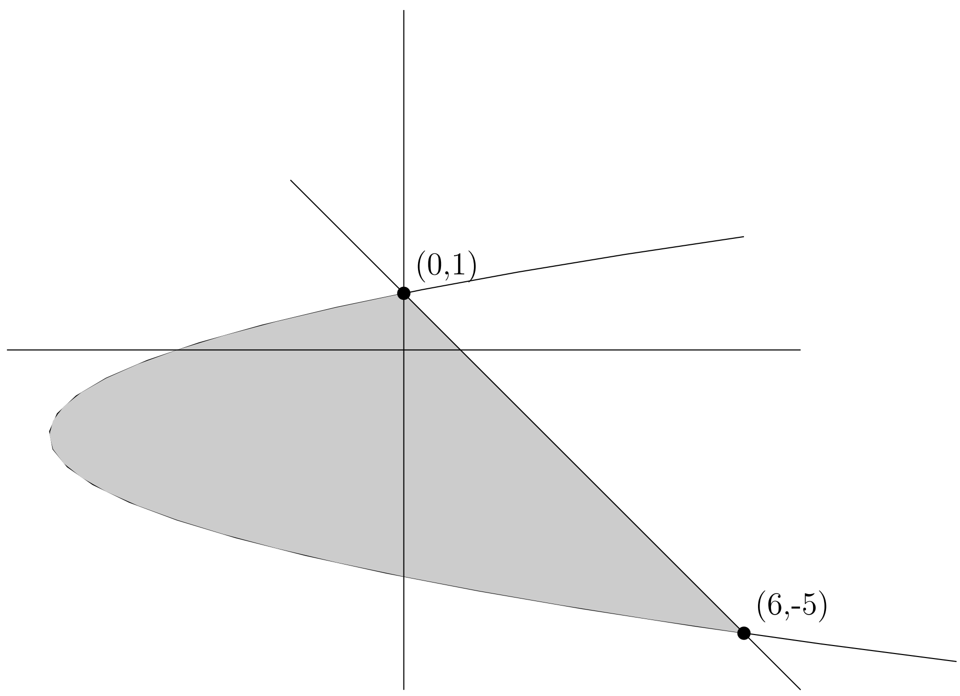
$$\begin{aligned}
f_x &= y - \frac{8}{x^2} \quad f_y = x - \frac{8}{y^2} \\
y &= \frac{8}{x^2},\ x = \frac{8}{y^2} \implies x^2 = \frac{64}{y^4} \implies \frac{1}{x^2} = \frac{y^2}{64} \\
y &= 8 \cdot \frac{y^4}{64} \\
8y &= y^4, y \neq 0 \text{ so } 8 = y^3 \text{ and } y = 2 \\
x &= \frac{8}{y^2} = \frac{8}{4} = 2 \text{ Crit pits: } x = 2, y = 2 \text{ only}
\end{aligned}$$

$$\begin{aligned}
f_{xx} &= \frac{16}{x^3} \quad f_{yy} = \frac{16}{y^3} \quad f_{xy} = 1 = f_{yx} \\
D &= f_{xx}f_{yy} - (f_{xy})^2 = \frac{256}{x^3y^3} - 1 \\
D(2,2) &= \frac{256}{8 \cdot 8} - 1 = 3 > 0 \text{ so local extremem and} \\
f_{xx}(2,2) &= \frac{16}{8} = 2 > 0 \text{ so local min at } (2,2)
\end{aligned}$$

Name: ______________________ Student #: ______________

B3. *[10 marks]*
Find the area of the finite region bounded by the graphs of $x = y^2 + 3y - 4$ and $y = 1 - x$.

$$\begin{aligned} x &= 1 - y \\ 1 - y &= y^2 + 3y - 4 \\ 0 &= y^2 + 4y - 5 \\ 0 &= (y+5)(y-1) \\ y &= -5 \;\; y = 1 \\ x &= 6 \;\; x = 0 \end{aligned}$$



$$\begin{aligned} A &= \int_{-5}^{1} \left[(1-y) - (y^2 + 3y - 4)\right] dy \\ &= \int_{-5}^{1} (5 - 4y - y^2))dy \\ &= \left[5y - 2y^2 - \frac{y^3}{3}\right]_{-5}^{1} \\ &= (5 - 2 - \frac{1}{3}) - (-25 - 50 + \frac{125}{3}) \\ &= 3 - \frac{1}{3} + 75 - \frac{125}{3} \\ &= 78 - \frac{126}{3} \\ &= 78 - 42 = 36 \end{aligned}$$

Name: ____________________ Student #: ____________________

B4. *[10 marks]*

(a) *[7 marks]*
If $y = f(x)$ is the most general solution to the differential equation

$$\frac{dy}{dx} = \frac{3y}{x} \quad \text{for } x, y > 0$$

find $f(x)$.

$$\begin{aligned}
\frac{dy}{y} &= 3\frac{dx}{x} \\
\ln y &= 3\ln x + c \quad (\text{no } |x|, |y| \text{ necessary since } x > 0, y > 0) \\
y &= e^{3\ln x + c} = e^{3\ln x} e^c \\
y &= Ax^3
\end{aligned}$$

(b) *[3 marks]*
What is the answer if $f(1) = 2$?

$$\begin{aligned}
2 &= A \cdot 1^3 \\
\text{So } A &= 2 \\
\text{and } y &= 2x^3
\end{aligned}$$

Name: ______________________________ Student #: ______________________

B5. *[11 marks]*
Evaluate

(a) *[5 marks]*
$\int x^2 \ln x dx$
Let $u = \ln x$, $dv = x^2 dx$
$du = \dfrac{dx}{x}$, $v = \dfrac{x^3}{3}$

$$\begin{aligned}\int x^2 \ln x dx &= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx\\ &= \frac{x^3}{x}\ln x - \frac{1}{9}x^3 + C\end{aligned}$$

(b) *[6 marks]*
$\int \dfrac{3x+1}{x(x+1)^2} dx$

$$\begin{aligned}\frac{3x+1}{x(x+1)^2} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\\ A(x+1)^2 + Bx(x+1) + Cx &= 3x+1\\ x = 0 &\implies A = 1\\ x = -1 \implies -C = -2 &\implies C = 2\\ x = 1 \text{ say } &\implies 4A + 2B + C = 4\\ 4 + 2B + 2 = 4 &\text{ so } B = -1\end{aligned}$$

$$\begin{aligned}\int \frac{3x+1}{x(x+1)^2} dx &= \int \left[\frac{1}{x} - \frac{1}{x+1} + \frac{2}{(x+1)^2}\right] dx\\ &= \ln|x| - \ln|x+1| - \frac{2}{x+1} + C\end{aligned}$$

Bonus: *[extra 5 marks possible]*
$\int_1^\infty \dfrac{3x+1}{x(x+1)^2} dx$

$$\begin{aligned}\lim_{R\to\infty}\int_1^R \frac{3x+1}{x(x+1)^2} dx &= \lim_{R\to\infty}\left[\ln|x| - \ln|x+1| - \frac{2}{x+1}\right]_1^R \text{ from (b)}\\ &= \lim_{R\to\infty}\left(\ln\frac{R}{R+1} - \frac{2}{R+1}\right) - (\ln 1 - \ln 2 - 1)\end{aligned}$$

But $\dfrac{R}{R+1} \to 1$ so $\ln\dfrac{R}{R+1} \to 0$ and $\dfrac{2}{R+1} \to 0$
The answer is $\ln 2 + 1$.