FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2015

MAT 133Y1Y
Calculus and Linear Algebra for Commerce

Duration: 3 hours
Examiners: A. Igelfeld
P. Kergin
L. Shorser
J. Tate

LEAVE BLANK	
Question	Mark
MC/45	
B1/10	
B2/10	
B3/12	
B4/11	
B5/12	
TOTAL	

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

NOTE:

1. **Aids Allowed:** A non-graphing calculator, with empty memory, to be supplied by student.
2. **Instructions:** Fill in the information on this page, and make sure your test booklet contains 14 pages.
3. This exam consists of 15 multiple choice questions, and 5 written-answer questions. For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the front page** with your pencil. Each correct answer is worth 3 marks; a question left blank, or an incorrect answer or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.
4. Put your name and student number on each page of this examination.

ANSWER BOX FOR PART A					
Circle the correct answer					
1.	**A.**	**B.**	**C.**	**D.**	**E.**
2.	**A.**	**B.**	**C.**	**D.**	**E.**
3.	**A.**	**B.**	**C.**	**D.**	**E.**
4.	**A.**	**B.**	**C.**	**D.**	**E.**
5.	**A.**	**B.**	**C.**	**D.**	**E.**
6.	**A.**	**B.**	**C.**	**D.**	**E.**
7.	**A.**	**B.**	**C.**	**D.**	**E.**
8.	**A.**	**B.**	**C.**	**D.**	**E.**
9.	**A.**	**B.**	**C.**	**D.**	**E.**
10.	**A.**	**B.**	**C.**	**D.**	**E.**
11.	**A.**	**B.**	**C.**	**D.**	**E.**
12.	**A.**	**B.**	**C.**	**D.**	**E.**
13.	**A.**	**B.**	**C.**	**D.**	**E.**
14.	**A.**	**B.**	**C.**	**D.**	**E.**
15.	**A.**	**B.**	**C.**	**D.**	**E.**

Record your answers on the front page

PART A. MULTIPLE CHOICE

1. *[3 marks]*
A bond has a face value of \$1000, 13 semi-annual coupons remaining with an annual coupon rate of 4%, and an annual yield rate of 2.4%. Its current price is closest to

   A. \$ 982.47

   B. \$1079.99

   C. \$ 955.78

   D. \$1176.88

   E. \$1095.76

2. *[3 marks]*
If

$$\begin{array}{rcrcrcl} x & - & 2y & & & = & 5 \\ 2x & & & + & 3z & = & 6 \\ & & 3y & + & 2z & = & 7 \end{array}$$

then $y =$

   A. 31

   B. 29

   C. 37

   D. 23

   E. no value, since the system has no solution

Record your answers on the front page

3. *[3 marks]*
If $2x^3 + x^2y + y^3 = 4$ defines y as a function of x, then when $x = 1$ and $y = 1$, $\dfrac{dy}{dx} =$
   A. -2
   B. -3
   C. -1
   D. -4
   E. -6

4. *[3 marks]*
Let $f(x) = x^3 - 9x^2 + 15x$ be defined **only** on the interval $[0, 7]$, that is, for $0 \leq x \leq 7$. Then which one of the following is true?
   A. f takes on its absolute maximum and minimum values at the endpoints.
   B. f takes on its absolute maximum value at two points.
   C. f has no absolute maximum or minimum.
   D. f takes on its absolute minimum value at an endpoint.
   E. f takes on its absolute minimum value at two points.

5. *[3 marks]*

The curve $y = \dfrac{\ln x}{x}$ has a point of inflection

A. nowhere

B. at $x = e$

C. at $x = e^{3/2}$

D. at $x = 1$

E. at $x = e^2$

6. *[3 marks]*

$\displaystyle\lim_{x\to 1} \frac{1}{1-x} + \frac{\ln x}{(1-x)^2} =$

A. -1

B. $\dfrac{1}{2}$

C. 1

D. ∞

E. $-\dfrac{1}{2}$

Record your answers on the front page

7. *[3 marks]*
If $f(x) = \int_1^x e^{\left(\frac{2}{t}-1\right)} dt$, then $f'(1) =$
 A. 1
 B. e
 C. 0
 D. $e^2 - 1$
 E. $-2e$

8. *[3 marks]*
If $f(x) = x + 2$, then the average value of f on the interval $[-1, 1]$ is
 A. 2
 B. 1
 C. 0
 D. -1
 E. -2

9. *[3 marks]*
$$\int_0^1 x^4\sqrt{x^5+1}\,dx =$$

A. $\dfrac{\sqrt{2}-1}{3}$

B. $\dfrac{\sqrt{2}}{3}$

C. $\dfrac{4\sqrt{2}}{15}$

D. $\dfrac{2(2\sqrt{2}-1)}{15}$

E. $\dfrac{2\sqrt{2}-1}{3}$

10. *[3 marks]*
$$\int_0^1 \frac{x^2+1}{x+1}\,dx =$$

A. $\ln 2$

B. 1

C. $1+2\ln 2$

D. $\dfrac{3}{2}$

E. $-\dfrac{1}{2}+2\ln 2$

Record your answers on the front page

11. *[3 marks]*
Let $f(x, y, z) = 5e^{5x^2-3y^4+2z}$. Then $f_z =$

A. $10e^{5x^2-3y^4+2z}$

B. $50xe^{5x^2-3y^4+2z}$

C. $5e^{5x^2-3y^4+2z}$

D. $e^{10x-12y^3+2}$

E. $e^{5x^2-3y^4+2}$

12. *[3 marks]*
Given the production function

$$f(L, K) = \sqrt{LK} + L^2 + 4K$$

where L is labour and K is capital, the marginal productivity with respect to labour when $L = 400$ and $K = 1,000,000$ is

A. $4,001,800$

B. 829

C. $25,800$

D. $4,180,000$

E. 825

13. *[3 marks]*

If $z = 5ye^x - x\ln y$, then $\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial^2 z}{\partial y^2} =$

A. $5e^x - \ln y$

B. 0

C. $5e^x - \dfrac{1}{y} - \dfrac{x}{y^2}$

D. $5 - \ln y$

E. $-\dfrac{1}{y} + \dfrac{x}{y^2}$

14. *[3 marks]*

The equation

$$s = 2t^2r + 3r^3s - 4ts^2$$

defines s as a function of r and t near the point $r = s = t = 1$. When $r = s = t = 1$, $\dfrac{\partial s}{\partial t} =$

A. -5

B. $\dfrac{2}{3}$

C. -1

D. 6

E. 0

Name: ______________________________ Student #: ______________

Record your answers on the front page

15. *[3 marks]*
Let p_A and p_B be the prices of product A and product B respectively and q_A and q_B be their respective quantities. The demand functions for each are given by:

$$q_A = 2p_A^2 p_B + k p_B \qquad\qquad q_B = e^{3p_A + p_B}$$

These two products are competitive at all positive prices

A. for all values of k.

B. for $k < 0$ only.

C. for $k \geq 0$ only.

D. for no values of k.

E. for $k > -2$ only.

Name: ____________________ Student #: ____________________

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[10 marks]*
A $\$600,000$ mortgage with monthly payments is amortized over 10 years.

(a) *[5 marks]*
Find the amount of each payment if interest is 8% compounded semiannually.

(b) *[5 marks]*
Immediately after the 5th year of the mortgage, the interest rate changes to 6% compounded semiannually. Find the new monthly payment required if the mortgage is still to be repaid in a total of 10 years.

Name: ____________________ Student #: ____________________

B2. *[10 marks]*

BigBoxCompany wants to choose how many televisions to keep in stock constantly throughout the year in its warehouse so as to minimize its annual cost. The annual cost C of keeping q televisions constantly in stock is given by

$$C(q) = 5q + \frac{200,000}{q} + 1000$$

for $q > 0$.

(a) *[5 marks]*

What is the minimum cost if $0 < q \leq 100$? (Justify your answer.)

(b) *[5 marks]*

The warehouse decides to increase its capacity. If the annual cost function remains the same no matter what they do, how much capacity would they need to have the absolute minimum cost? (Justify your answer.)

B3. *[12 marks]*

Evaluate the following integrals

(a) *[6 marks]*

$$\int_1^{\infty} x^2 e^{-x}\, dx$$

(b) *[6 marks]*

$$\int \frac{3x-2}{x^3+2x^2}\, dx$$

Name: ______________________ Student #: ______________

B4. *[11 marks]*

(a) *[5 marks]*

The marginal cost of producing widgets is given by $\frac{dC}{dq} = \frac{1}{q^2} + e^{1-q}$ where q is the number of widgets and C is the total cost function. Given that $C(1) = 7$, find the total cost (to 2 decimal places) of producing $q = 5$.

(b) *[6 marks]*

Find all functions y explicitly in terms of x such that $\frac{dy}{dx} = \sqrt{x}(1-x)y$ for $x > 0$.

Name: ____________________ Student #: ________________

B5. *[12 marks]*

(a) *[6 marks]*
Find and classify the critical points of

$$f(x,y) = x^2 - xy + y^2 - 2x + y$$

(b) *[6 marks]*
Use **Lagrange multipliers** to find all critical points of

$$f(x,y) = x^{1/2}y^{3/4}$$

subject to the constraint

$$x + 2y = 30$$

Note: **No** marks will be assigned to another method of solution.

Name: ______________________ Student #: ______________

PART A. MULTIPLE CHOICE

1. *[3 marks]*
A bond has a face value of \$1000, 13 semi-annual coupons remaining with an annual coupon rate of 4%, and an annual yield rate of 2.4%. Its current price is closest to

   A. \$ 982.47

   B. \$1079.99

   C. \$ 955.78

   D. \$1176.88

   E. \$1095.76

$P = V(1+i)^{-n} + rVa_{\overline{n}|i}$
$n = 13 \quad V = 1000 \quad r = .02 \quad i = .012$
$P = 1000(1.012)^{-13} + 20a_{\overline{13}|.012}$
$\quad = \$1095.76$ **E**

2. *[3 marks]*
If

$$\begin{array}{rcrcrcr} x & - & 2y & & & = & 5 \\ 2x & & & + & 3z & = & 6 \\ & & 3y & + & 2z & = & 7 \end{array}$$

then $y =$

   A. 31

   B. 29

   C. 37

   D. 23

   E. no value, since the system has no solution

$$\begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left(\begin{array}{rrr|r} 1 & -2 & 0 & 5 \\ 2 & 0 & 3 & 6 \\ 0 & 3 & 2 & 7 \end{array}\right) \end{array} \xrightarrow{R_2 \to -2R_1 + R_2} \left(\begin{array}{rrr|r} 1 & -2 & 0 & 5 \\ 0 & 4 & 3 & -4 \\ 0 & 3 & 2 & 7 \end{array}\right)$$

$$\xrightarrow[R_3 \to -3R_2 + R_3]{R_2 \to \frac{1}{4}R_2} \left(\begin{array}{rrr|r} 1 & -1 & 0 & 5 \\ 0 & 1 & 3/4 & -1 \\ 0 & 0 & -1/4 & 10 \end{array}\right) \xrightarrow{R_4 \to -4R_4} \left(\begin{array}{rrr|r} 1 & -1 & 0 & 5 \\ 0 & 1 & 3/4 & -1 \\ 0 & 0 & 1 & -40 \end{array}\right)$$

$$\xrightarrow{R_3 \to -3/4R_3 + R_2} \left(\begin{array}{ccc|r} & \text{irrelevant} & & \\ 0 & 1 & 0 & 29 \\ 0 & 0 & 1 & -40 \end{array}\right)$$

so $y = 29$ **B**

or: back substitution $z = -40 \qquad y = -1 + \frac{3}{4}z = -1 + \frac{3}{4}(40) = 29$ **B**

Name: ______________________ Student #: ______________

3. *[3 marks]*
If $2x^3 + x^2y + y^3 = 4$ defines y as a function of x, then when $x = 1$ and $y = 1$, $\dfrac{dy}{dx} =$

A. -2

B. -3

C. -1

D. -4

E. -6

$6x^2 + 2xy + x^2\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$

At $(1,1)$: $6 + 2 + \dfrac{dy}{dx} + 3\dfrac{dy}{dx} = 0$

$4\dfrac{dy}{dx} = -8$ and $\dfrac{dy}{dx} = -2$ **A**

or: $(x^2 + 3y^2)\dfrac{dy}{dx} = -(6x^2 + 2xy)$

$$\frac{dy}{dx} = \frac{-(6x^2 + 2xy)}{x^2 + 3y^2} = -2 \text{ at } (1,1).$$

4. *[3 marks]*
Let $f(x) = x^3 - 9x^2 + 15x$ be defined **only** on the interval $[0,7]$, that is, for $0 \le x \le 7$. Then which one of the following is true?

A. f takes on its absolute maximum and minimum values at the endpoints.

B. f takes on its absolute maximum value at two points.

C. f has no absolute maximum or minimum.

D. f takes on its absolute minimum value at an endpoint.

E. f takes on its absolute minimum value at two points.

Since f is cont. on the closed interval $[0,7]$, it must take on an absolute min value **and** an absolute max value (C is false).

$f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x-5)(x-1)$ crit. pts. are $x = 1$ and $x = 5$.

The candidates for max and min:

$x = 0$	$f(x) =$	0
$x = 1$	$f(x) =$	7 max
$x = 5$	$f(x) =$	-25 min (A and D are false)
$x = 7$	$f(x) =$	7 max

E is false because the min is at $x = 5$ **only**.

B is true: max at $x = 1$ and $x = 7$.

Name: ________________________ Student #: ________________

5. *[3 marks]*
The curve $y = \dfrac{\ln x}{x}$ has a point of inflection

A. nowhere

B. at $x = e$

C. at $x = e^{3/2}$

D. at $x = 1$

E. at $x = e^2$

$$\frac{dy}{dx} = \frac{x\frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1-\ln x)2x}{x^4} = \frac{-3+2\ln x}{x^3}$$

$$\frac{d^2y}{dx^2} = 0 \text{ when } -3+2\ln x = 0$$

$$\ln x = \frac{3}{2}$$

$$x = e^{3/2}$$

$\dfrac{\ln x}{x}$ undefined for $x \le 0$

	y''	y
$(0, e^{3/2})$	$-$	conc. down
$(e^{3/2}, \infty)$	$+$	conc. up

$x = e^{3/2}$ is p.o.i **C**

6. *[3 marks]*
$\displaystyle\lim_{x\to 1} \frac{1}{1-x} + \frac{\ln x}{(1-x)^2} =$

A. -1

B. $\dfrac{1}{2}$

C. 1

D. ∞

E. $-\dfrac{1}{2}$

$$\lim_{x\to 1} \frac{1-x+\ln x}{(1-x)^2} \qquad \frac{0}{0} \text{ L'Hôp}$$

$$= \lim_{x\to 1} \frac{-1+\frac{1}{x}}{2(1-x)(-1)} \qquad \frac{0}{0} \text{ L'Hôp}$$

$$= \lim_{x\to 1} \frac{-\frac{1}{x^2}}{+2} = -\frac{1}{2} \qquad \textbf{E}$$

$$\text{or: } \lim_{x\to 1} \frac{-1+\frac{1}{x}}{2(1-x)(-1)} = \lim_{x\to 1} \frac{1-x}{2(1-x)(-1)} = -\frac{1}{2}$$

7. *[3 marks]*
If $f(x) = \int_1^x e^{\left(\frac{2}{t}-1\right)}\,dt$, then $f'(1) =$

A. 1

B. e

C. 0

D. $e^2 - 1$

E. $-2e$

$f'(x) = e^{2/x-1}$
$f'(1) = e^{2/1-1} = e$ **B**

8. *[3 marks]*
If $f(x) = x + 2$, then the average value of f on the interval $[-1, 1]$ is

A. 2

B. 1

C. 0

D. -1

E. -2

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2}\int_{-1}^{1} (x+2)\,dx$$

$$= \frac{1}{2}\frac{(x+2)^2}{2}\bigg|_{-1}^{1}$$

$$= \frac{1}{4}(9-1) = 2 \qquad \textbf{A}$$

Name: ______________________ Student #: ______________

9. *[3 marks]*
$$\int_0^1 x^4\sqrt{x^5+1}\,dx =$$

A. $\dfrac{\sqrt{2}-1}{3}$

B. $\dfrac{\sqrt{2}}{3}$

C. $\dfrac{4\sqrt{2}}{15}$

D. $\dfrac{2(2\sqrt{2}-1)}{15}$

E. $\dfrac{2\sqrt{2}-1}{3}$

Let $u = x^5+1 \quad du = 5x^4dx$
$x = 0 \quad u = 1$
$x = 1 \quad u = 2$

$$\frac{1}{5}\int_1^2 \sqrt{u}\,du = \frac{1}{5}u^{3/2}\frac{2}{3}\bigg|_1^2$$
$$= \frac{2}{15}\left(2^{3/2}-1\right) \qquad \textbf{D}$$

10. *[3 marks]*
$$\int_0^1 \frac{x^2+1}{x+1}\,dx =$$

A. $\ln 2$

B. 1

C. $1+2\ln 2$

D. $\dfrac{3}{2}$

E. $-\dfrac{1}{2}+2\ln 2$

$$\begin{array}{r l}
 & x - 1 \\
x+1 \, \big) & x^2+1 \\
 & \underline{x^2+x} \\
 & -x+1 \\
 & \underline{-x-1} \\
 & 2
\end{array}$$

$$\int_0^1 \left[(x-1)+\frac{2}{x+1}\right]\,dx$$
$$= \left[\frac{x^2}{2}-x+2\ln|x+1|\right]\bigg|_0^1$$
$$= \left[\frac{1}{2}-1+2\ln 2\right]-[0-0+2\ln 1]$$
$$= -\frac{1}{2}+2\ln 2 \qquad \textbf{E}$$

Name: ____________________ Student #: ____________________

11. *[3 marks]*
Let $f(x, y, z) = 5e^{5x^2-3y^4+2z}$. Then $f_z =$

A. $10e^{5x^2-3y^4+2z}$

B. $50xe^{5x^2-3y^4+2z}$

C. $5e^{5x^2-3y^4+2z}$

D. $e^{10x-12y^3+2}$

E. $e^{5x^2-3y^4+2}$

$f_z = 5e^{5x^2-3y^4+2z} \cdot 2$ **A**

12. *[3 marks]*
Given the production function

$$f(L, K) = \sqrt{LK} + L^2 + 4K$$

where L is labour and K is capital, the marginal productivity with respect to labour when $L = 400$ and $K = 1,000,000$ is

A. $4,001,800$

B. 829

C. $25,800$

D. $4,180,000$

E. 825

$$\begin{aligned} \frac{\partial F}{\partial L} &= \frac{\sqrt{K}}{2\sqrt{L}} + 2L \\ &= \frac{\sqrt{1,000,000}}{2\sqrt{400}} + 2 \cdot 400 \\ &= \frac{1000}{2 \cdot 20} + 800 \\ &= 825 \end{aligned}$$ **E**

Name: ______________________ Student #: ______________

13. *[3 marks]*

If $z = 5ye^x - x\ln y$, then $\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial^2 z}{\partial y^2} =$

A. $5e^x - \ln y$

B. 0

C. $5e^x - \dfrac{1}{y} - \dfrac{x}{y^2}$

D. $5 - \ln y$

E. $-\dfrac{1}{y} + \dfrac{x}{y^2}$

$\dfrac{\partial z}{\partial y} = 5e^x - \dfrac{x}{y}$

So $\dfrac{\partial^2 z}{\partial x \partial y} = 5e^x - \dfrac{1}{y}$

and $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{x}{y^2}$

$\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial^2 z}{\partial y^2} = 5e^x - \dfrac{1}{y} - \dfrac{x}{y^2}$ **C**

14. *[3 marks]*

The equation

$$s = 2t^2 r + 3r^3 s - 4ts^2$$

defines s as a function of r and t near the point $r = s = t = 1$. When $r = s = t = 1$, $\dfrac{\partial s}{\partial t} =$

A. -5

B. $\dfrac{2}{3}$

C. -1

D. 6

E. 0

$\dfrac{\partial s}{\partial t} = 4tr + 3r^3\dfrac{\partial s}{\partial t} - 4s^2 - 8ts\dfrac{\partial s}{\partial t}$

$\dfrac{\partial s}{\partial t} = 4 + 3\dfrac{\partial s}{\partial t} - 4 + 8\dfrac{\partial s}{\partial t}$

$-10\dfrac{\partial s}{\partial t} = 0$ $\dfrac{\partial s}{\partial t} = 0$ **E**

Name: ______________________________ Student #: ____________________

15. *[3 marks]*
Let p_A and p_B be the prices of product A and product B respectively and q_A and q_B be their respective quantities. The demand functions for each are given by:

$$q_A = 2p_A^2 p_B + kp_B \qquad\qquad q_B = e^{3p_A+p_B}$$

These two products are competitive at all positive prices

A. for all values of k.

B. for $k < 0$ only.

C. for $k \geq 0$ only.

D. for no values of k.

E. for $k > -2$ only.

$\dfrac{\partial q_A}{\partial p_B} = 2p_A^2 + k > 0$ for all $p_A > 0$ as long as $k \geq 0$

$\dfrac{\partial q_B}{\partial p_A} = 3e^{3p_A+p_B} > 0$ no matter what

Competitive at all positive prices requires $\dfrac{\partial q_A}{\partial p_B}$ and $\dfrac{\partial q_A}{\partial p_B} > 0$ for all p_A and p_B positive so **C**

Name: ____________________ Student #: ________________

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[10 marks]*

A $600,000 mortgage with monthly payments is amortized over 10 years.

(a) *[5 marks]*

Find the amount of each payment if interest is 8% compounded semiannually.

$(1.04)^2 = (1+i)^{12} \qquad (1.04)^{-20} = (1+i)^{-120}$

$$600,000 = Ra_{\overline{120}|i}$$

$$R = \frac{600,000}{a_{\overline{120}|i}} = \frac{600,000i}{1-(1+i)^{-120}} = \frac{600,000[(1.04)^{1/6}-1]}{1-(1.04)^{-20}}$$

$\boxed{\text{R=7238.45}}$

(b) *[5 marks]*

Immediately after the 5th year of the mortgage, the interest rate changes to 6% compounded semiannually. Find the new monthly payment required if the mortgage is still to be repaid in a total of 10 years.

$$\begin{aligned}\text{Principal outstanding (with 60 payments remaining)} &= Ra_{\overline{60}|i} \\ &= \frac{600,000}{a_{\overline{120}|i}} a_{\overline{60}|i} \\ &= 600,000 \frac{[1-(1+i)^{-60}]/i}{[1-(1+i)^{-120}]/i} \\ &= 600,000 \frac{[1-(1.04)^{-10}]}{[1-(1.04)^{-20}]} = \$358,088.24\end{aligned}$$

P.O.$= R'a_{\overline{60}|i'} \qquad (1.03)^2 = (1+i')^{12} : (1.03)^{-10} = (1+i')^{-60}$

$$R' = \frac{358,088.34}{a_{\overline{60}|i'}} = 358,088.34 \frac{[(1.03)^{1/6}-1]}{[1-(1.03)^{-10}]}$$

$\boxed{\text{R'=\$6910.59}}$

Name: ______________________ Student #: ______________

B2. *[10 marks]*
BigBoxCompany wants to choose how many televisions to keep in stock constantly throughout the year in its warehouse so as to minimize its annual cost. The annual cost C of keeping q televisions constantly in stock is given by

$$C(q) = 5q + \frac{200,000}{q} + 1000$$

for $q > 0$.

(a) *[5 marks]*
What is the minimum cost if $0 < q \leq 100$? (Justify your answer.)

$$\frac{dC}{dq} = 5 - \frac{200,000}{q^2} = 0 \text{ when } q^2 = \frac{200,000}{5} = 40,000 \text{ i.e. when } q = 200$$

	$\frac{dC}{dq}$	C
$(0, 200)$	$-$	decreasing
$(200, \infty)$	$+$	increasing

On the interval $(0, 100]$, C is decreasing, so its minimum value is at $q = 100$ when $C = 5 \cdot 100 + \frac{200,000}{100} + 1000$

$\boxed{\text{C=3,500}}$

(b) *[5 marks]*
The warehouse decides to increase its capacity. If the annual cost function remains the same no matter what they do, how much capacity would they need to have the absolute minimum cost? (Justify your answer.)

Since C decreases all the way to $q = 200$, and increases forever afterward, the minimum cost will be at $\boxed{\text{q=200}}$.

Name: ______________________________ Student #: ______________

B3. *[12 marks]*

Evaluate the following integrals

(a) *[6 marks]*

$$\int_1^\infty x^2 e^{-x}\, dx$$

$$\int_1^\infty x^2 e^{-x}\, dx = \lim_{R\to\infty} \int_1^R x^2 e^{-x}\, dx \qquad u = x^2 \qquad dv = e^{-x}dx$$

$$du = 2xdx \qquad v = -e^{-x}$$

$$= \lim_{R\to\infty} \left[-x^2 e^{-x}\Big|_1^R + 2\int_1^R x e^{-x}\, dx \right]$$

$$u = x \qquad dv = e^{-x}dx$$

$$du = dx \qquad v = -e^{-x}$$

$$= \lim_{R\to\infty} \left[-R^2 e^{-R} + e^{-1} + 2\left\{ -xe^{-x}\Big|_1^R + \int_1^R e^{-x} \right\} \right]$$

$$= \lim_{R\to\infty} \left[-R^2 e^{-R} + e^{-1} + 2\left\{ -Re^{-R} + e^{-1} - e^{-x}\Big|_1^R \right\} \right]$$

$$= \lim_{R\to\infty} \left[-R^2 e^{-R} + e^{-1} - 2Re^{-R} + 2e^{-1} - 2e^{-R} + 2e^{-1} \right]$$

$$= 5e^{-1} - \lim_{R\to\infty} e^{-R}(R^2 + 2R + 2)$$

but $\lim_{R\to\infty} \dfrac{R^2+2R+2}{e^R} = 0$ by two applications of L'Hópital

so $= \boxed{5e^{-1}}$

(b) *[6 marks]*

$$\int \frac{3x-2}{x^3+2x^2}\, dx$$

$$\frac{3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$Ax(x+2) + B(x+2) + Cx^2 = 3x - 2$

$x = 0 \quad \Rightarrow 2B = -2 \quad \boxed{\text{B=-1}}$

$x = -2 \quad \Rightarrow 4C = -8 \quad \boxed{\text{C=-2}}$

$x = -1$ say $\Rightarrow -A + B + C = -5$

$-A + (-1) + (-2) = -5$

$-A = 2 \quad \boxed{\text{A=2}}$

$$\int \left[\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+2} \right] dx$$

$$= \boxed{2\ln|x| + \frac{1}{x} - 2\ln|x+2| + C}$$

or $\boxed{2\ln\left|\dfrac{x}{x+2}\right| + \dfrac{1}{x}} + C$

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B4. *[11 marks]*

(a) *[5 marks]*

The marginal cost of producing widgets is given by $\frac{dC}{dq} = \frac{1}{q^2} + e^{1-q}$ where q is the number of widgets and C is the total cost function. Given that $C(1) = 7$, find the total cost (to 2 decimal places) of producing $q = 5$.

$C = \int \left(\frac{1}{q^2} + e^{1-q}\right) dq = -\frac{1}{q} - e^{1-q} + K$

$7 = -1 - e^0 + K \Rightarrow K = 9$

$C = -\frac{1}{q} - e^{1-q} + 9$

$C(5) = -\frac{1}{5} - e^{-4} + 9 = 8.8 - e^{-4} = \boxed{8.78}$ to 2 places

(b) *[6 marks]*

Find all functions y explicitly in terms of x such that $\frac{dy}{dx} = \sqrt{x}(1-x)y$ for $x > 0$.

$\frac{dy}{y} = \sqrt{x}(1-x) = x^{-1/2} - x^{3/2}$

$\ln|y| = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + K$

$|y| = e^K e^{\frac{2}{3}e^{3/2} - \frac{2}{5}x^{5/2}}$

and if $D = \pm e^K$

$\boxed{y = De^{\frac{2}{3}e^{3/2} - \frac{2}{5}x^{5/2}}}$, D arbitrary

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B5. *[12 marks]*

(a) *[6 marks]*
Find and classify the critical points of

$$f(x, y) = x^2 - xy + y^2 - 2x + y$$

$$\left.\begin{array}{ll} f_x = \ \ 2x - \ \ y - 2 = 0 & 2x - \ \ y = \ \ 2 \\ f_y = -x + 2y + 1 = 0 & -x + 2y = -1 \end{array}\right\} x = 1,\ y = 0$$

$\boxed{\text{The only critical point is } x = 1,\ y = 0}$

$f_{xx} = 2 \qquad f_{yy} = 2 \qquad f_{xz} = -1$

$D = f_{xx}f_{yy} - (f_{xy})^2 = 4 - 1 = 3 > 0$ so local extremum and $f_{xx} > 0$ so $\boxed{\text{local min}}$

$\boxed{\text{(1,0) is a local min}}$

(b) *[6 marks]*
Use **Lagrange multipliers** to find all critical points of

$$f(x, y) = x^{1/2}y^{3/4}$$

subject to the constraint

$$x + 2y = 30$$

Note: **No** marks will be assigned to another method of solution.

$\mathscr{L} = x^{1/2}y^{3/4} - \lambda(x + 2y - 30)$

$\mathscr{L}_x = \dfrac{1}{2}x^{-1/2}y^{3/4} - \lambda = 0$

$\mathscr{L}_y = \dfrac{3}{4}x^{1/2}y^{-1/4} - 2\lambda = 0$

$\mathscr{L}_\lambda = 0 \Rightarrow x + 2y = 30$

$x^{-1/2}y^{3/4} = 2\lambda$

$x^{1/2}y^{-1/4} = \dfrac{8\lambda}{3}$

Dividing the second equation by the first

$\dfrac{x^{1/2}y^{-1/4}}{x^{-1/2}y^{3/4}} = \dfrac{4}{3} \Rightarrow \dfrac{x}{y} = \dfrac{4}{3} \Rightarrow x = \dfrac{4y}{3}$

$\dfrac{4y}{3} + 2y = 30 \Rightarrow 10y = 90 \Rightarrow y = 9 \Rightarrow x = 12$

$\boxed{\text{x=12, \quad y=9}}$