FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2014

MAT 133Y1Y
Calculus and Linear Algebra for Commerce

Duration: 3 hours
Examiners: A. Igelfeld
P. Kergin
L. Shorser
J. Tate

LEAVE BLANK	
Question	Mark
MC/45	
B1/11	
B2/11	
B3/13	
B4/8	
B5/12	
TOTAL	

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

NOTE:

1. **Aids Allowed:** A non-graphing calculator, with empty memory, to be supplied by student.
2. **Instructions:** Fill in the information on this page, and make sure your test booklet contains 14 pages.
3. This exam consists of 15 multiple choice questions, and 5 written-answer questions. For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the front page** with your pencil. Each correct answer is worth 3 marks; a question left blank, or an incorrect answer or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.
4. Put your name and student number on each page of this examination.

ANSWER BOX FOR PART A

Circle the correct answer

1.	**A.**	**B.**	**C.**	**D.**	**E.**
2.	**A.**	**B.**	**C.**	**D.**	**E.**
3.	**A.**	**B.**	**C.**	**D.**	**E.**
4.	**A.**	**B.**	**C.**	**D.**	**E.**
5.	**A.**	**B.**	**C.**	**D.**	**E.**
6.	**A.**	**B.**	**C.**	**D.**	**E.**
7.	**A.**	**B.**	**C.**	**D.**	**E.**
8.	**A.**	**B.**	**C.**	**D.**	**E.**
9.	**A.**	**B.**	**C.**	**D.**	**E.**
10.	**A.**	**B.**	**C.**	**D.**	**E.**
11.	**A.**	**B.**	**C.**	**D.**	**E.**
12.	**A.**	**B.**	**C.**	**D.**	**E.**
13.	**A.**	**B.**	**C.**	**D.**	**E.**
14.	**A.**	**B.**	**C.**	**D.**	**E.**
15.	**A.**	**B.**	**C.**	**D.**	**E.**

Record your answers on the front page

PART A. MULTIPLE CHOICE

1. *[3 marks]*
If z is used as the parameter in the solution set of the system

$$\begin{aligned} 2x + 3y + 5z &= 3 \\ 3x + 4y + 6z &= 5 \end{aligned}$$

then $x =$

A. $2 - 3z$

B. $3 + 2z$

C. $-2 + z$

D. $1 + 3z$

E. $-1 - 3z$

2. *[3 marks]*
If $f(x) = \dfrac{\sqrt{3x-5}}{x}$, then $f'(2) =$

A. 0

B. $-\dfrac{1}{2}$

C. 1

D. -1

E. $\dfrac{1}{2}$

3. *[3 marks]*
If $y(x)$ satisfies $y^4 + 1 = xy + x^2$ and $y = 1$ when $x = 1$, then when $x = 1$, $y' =$
   A. $-\dfrac{1}{2}$
   B. $\dfrac{1}{4}$
   C. -1
   D. $\dfrac{1}{2}$
   E. 1

4. *[3 marks]*
If $2x^y = e^2 y$, then when $(x, y) = (e, 2)$, $\dfrac{dy}{dx} =$
   A. $\dfrac{e}{4}$
   B. $-\dfrac{e}{2}$
   C. $-\dfrac{4}{e}$
   D. $2e$
   E. $-\dfrac{2}{e}$

5. *[3 marks]*
On the interval $[-2, 4]$, the function $f(x) = 2x^3 - 9x^2$ has

A. an absolute minimum at $x = \ \ 3$ and an absolute maximum at $x = 4$.

B. an absolute minimum at $x = -2$ and no absolute maximum.

C. an absolute minimum at $x = \ \ 3$ and an absolute maximum at $x = 0$.

D. an absolute minimum at $x = -2$ and an absolute maximum at $x = 0$.

E. an absolute minimum at $x = -2$ and an absolute maximum at $x = 4$.

6. *[3 marks]*
$$\lim_{x \to 1} \frac{x - 1 - \ln x}{x - 2\sqrt{x} + 1} =$$

A. $\frac{1}{2}$

B. 1

C. 2

D. ∞

E. 0

Record your answers on the front page

7. *[3 marks]*
If $x_1 = 0$ is used as a first estimate to approximate a root of $x^3 + x = 1$ by Newton's method, then the third estimate, x_3, equals

A. $\frac{3}{4}$

B. $\frac{2}{3}$

C. $\frac{5}{6}$

D. $\frac{1}{2}$

E. $\frac{5}{8}$

8. *[3 marks]*
$\int_1^e (\frac{1}{x} - \frac{1}{x^2})\, dx =$

A. $\frac{1}{e} - \frac{1}{e^2}$

B. $-\frac{1}{e^2} + \frac{2}{e^3}$

C. $\frac{1}{e} - 1$

D. $\frac{1}{e}$

E. 1

9. *[3 marks]*

$$\int_0^2 x^2\sqrt{1+x^3}\,dx =$$

A. $\sqrt{35}$

B. $\sqrt{6}$

C. $\frac{26}{3}$

D. $\frac{52}{9}$

E. 36

10. *[3 marks]*

Let $g(x) = \int_1^x \sqrt{2^t+1}\,dt$. Then $g'(2) =$

A. $\sqrt{5}$

B. $\sqrt{5}-\sqrt{2}$

C. $\frac{2\ln 2}{\sqrt{5}}$

D. $\sqrt{2^x+1}$

E. 0

11. *[3 marks]*
Let $f(x, y) = x^2y^3 + e^{xy}$. Then $f_y =$

A. $6xy^2 + xye^{xy}$

B. $3y^2 + e^x$

C. $3x^2y^2 + e^{xy}$

D. $3x^2y^2 + ye^{xy}$

E. $3x^2y^2 + xe^{xy}$

12. *[3 marks]*
If $y + xz^2 - x^2z^3 = 2$ defines z implicitly as a function of x and y near the point $x = 1$, $y = 2$, $z = 1$, then at that point $\dfrac{\partial z}{\partial y} =$

A. -1

B. 1

C. 0

D. -6

E. 5

13. *[3 marks]*
If $f(x,y) = x^2 e^{y^2}$ then $f_{xy}(1,1) =$

A. $4e$

B. $12e$

C. $6e$

D. $8e$

E. $2e$

14. *[3 marks]*
If $z = x^2 + xy + y^2$ where $x = 3t - 6$ and $y = t^2 + 2$ then when $t = 1$ $\quad \dfrac{dz}{dt} =$

A. 9

B. 3

C. -3

D. 15

E. 6

15. *[3 marks]*

The joint demand functions for the products A and B are given by:

$$q_A = \frac{200}{p_A\sqrt{p_B}} \qquad\qquad q_B = \frac{300}{p_B\sqrt[3]{p_A}}$$

Which of the following statements is true?

A. $\dfrac{\partial q_A}{\partial p_A} > 0$

B. $\dfrac{\partial q_B}{\partial p_B} > 0$

C. Products A and B are complementary

D. Products A and B are competitive

E. Products A and B are neither complementary nor competitive

Name: ______________________ Student #: ______________

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[11 marks]*

(a) *[4 marks]*
A 20 year mortgage for $\$500,000$ has monthly payments with interest at 4% compounded semiannually. Find the amount of each payment (to the nearest cent).

(b) *[3 marks]*
Find the principal outstanding (to the nearest cent) in the mortgage of question 1.(a), just after the 144th payment has been made.

(c) *[4 marks]*
What is the market price (to the nearest cent) of a $100 bond having 9 years until maturity and semiannual coupons, with annual coupon rate 6% and annual yield rate 5%?

Name: ______________________________ Student #: ______________

B2. *[11 marks]*

Consider the graphs of $x = y^2 + 1$ and $x = 4y + 1$

(a) *[2 marks]*
Find the points where those graphs intersect.

(b) *[5 marks]*
Express as an integral the finite area bounded by those graphs.

(c) *[4 marks]*
Find the area from part (b).

Name: ______________________ Student #: ______________

B3. *[13 marks]*

Evaluate the following integrals

(a) *[6 marks]*

$$\int_0^1 (2x+1)e^{2x}\, dx$$

(b) *[7 marks]*

$$\int_1^\infty \frac{1}{x^2(x+1)}\, dx$$

B4. *[8 marks]*

Assuming that $y > 0$, find an expression for y in terms of x if y satisfies the differential equation

$$\frac{dy}{dx} = xy$$

and $y = 3$ when $x = 0$. [Hint: This one's easy.]

Name: ______________________________ Student #: ______________

B5. *[12 marks]*

(a) *[6 marks]*
Find and classify the critical point(s) of

$$f(x, y) = 5x^2 - 2xy + 2y^2 - 10x + 2y$$

(b) *[6 marks]*
By using the method of Lagrange multipliers **only** find the critical points of the joint cost function

$$c(q_A, q_B) = q_A^2 - q_A q_B + \frac{3}{2} q_B^2 + 300$$

subject to the constraint

$$q_A + q_B = 700.$$

[Show all your work. No marks will be given for any other method.]

Name: ____________________ Student #: ____________________

PART A. MULTIPLE CHOICE

1. *[3 marks]*
If z is used as the parameter in the solution set of the system

$$\begin{aligned} 2x+3y+5z &= 3 \\ 3x+4y+6z &= 5 \end{aligned}$$

then $x =$

A. $2-3z$

B. $3+2z$

C. $-2+z$

D. $1+3z$

E. $-1-3z$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left(\begin{array}{ccc|c} 2 & 3 & 5 & 3 \\ 3 & 4 & 6 & 5 \end{array}\right) \xrightarrow[R_3\to -3R_1+R_2]{R_1\to\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 3/2 & 5/2 & 3/2 \\ 0 & -1/2 & -3/2 & 1/2 \end{array}\right) \xrightarrow{R_1\to R_1+3R_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & -1/2 & -1/2 & 1/2 \end{array}\right)$$

$x = 3+2z$, **B**

Or eliminate y directly:

$$\begin{aligned} 8x+12y+20z &= 12 \\ 9x+12y+18z &= 15 \\ \hline x-2z &= 3 \end{aligned}$$

$x = 3+2z$, **B**

2. *[3 marks]*
If $f(x) = \dfrac{\sqrt{3x-5}}{x}$, then $f'(2) =$

A. 0

B. $-\dfrac{1}{2}$

C. 1

D. -1

E. $\dfrac{1}{2}$

Quotient Rule: $f'(x) = \dfrac{x * \frac{1}{2\sqrt{3x-5}} * 3 - \sqrt{3x-5}}{x^2}$

$$f'(2) = \frac{\frac{2*3}{2\sqrt{1}} - \sqrt{1}}{4} = \frac{1}{2},$$ **E**

Name: ______________________________ Student #: ____________________

3. *[3 marks]*
If $y(x)$ satisfies $y^4 + 1 = xy + x^2$ and $y = 1$ when $x = 1$, then when $x = 1$, $y' =$

A. $-\frac{1}{2}$

B. $\frac{1}{4}$

C. -1

D. $\frac{1}{2}$

E. 1

$4y^3y' = y + xy' + 2x$. Substituting $x = 1, y = 1$,

$4y' = 1 + y' + 2$

$3y' = 3$

$y' = 1$, **E**

4. *[3 marks]*
If $2x^y = e^2y$, then when $(x, y) = (e, 2)$, $\frac{dy}{dx} =$

A. $\frac{e}{4}$

B. $-\frac{e}{2}$

C. $-\frac{4}{e}$

D. $2e$

E. $-\frac{2}{e}$

Taking ln of both sides:

$\ln 2 + y \ln x = 2 + \ln y$

$y' \ln x + \frac{y}{x} = \frac{1}{y}y'$

$y' \ln e + \frac{2}{e} = \frac{1}{2}y'$

$y' - \frac{1}{2}y' = -\frac{2}{e}$

$\frac{1}{2}y' = -\frac{2}{e}$

$y' = -\frac{4}{e}$, **C**

Name: ____________________ Student #: ____________________

5. *[3 marks]*
On the interval $[-2, 4]$, the function $f(x) = 2x^3 - 9x^2$ has

A. an absolute minimum at $x = \quad 3$ and an absolute maximum at $x = 4$.

B. an absolute minimum at $x = -2$ and no absolute maximum.

C. an absolute minimum at $x = \quad 3$ and an absolute maximum at $x = 0$.

D. an absolute minimum at $x = -2$ and an absolute maximum at $x = 0$.

E. an absolute minimum at $x = -2$ and an absolute maximum at $x = 4$.

$f'(x) = 6x^2 - 18x = 6x(x-3)$
Critical points are $x = 0$ and $x = 3$
Endpoints of closed intervals are $x = -2$ and $x = 4$
The continuous function f must have absolute masx and min among these 4 points.
$f(-2) = 2(-8) - 36 = -52$ **min** $x = -2$
$f(0) = 0$ **max** $x = 0$
$f(3) = 54 - 9 * 9 = -27$
$f(4) = 2 * 64 - 9 * 16 = -16,$ **D**

6. *[3 marks]*
$\lim_{x \to 1} \dfrac{x - 1 - \ln x}{x - 2\sqrt{x} + 1} =$

A. $\dfrac{1}{2}$

B. 1

C. 2

D. ∞

E. 0

$\dfrac{0}{0}$ L'Hopital

$= \lim_{x \to 1} \dfrac{1 - \frac{1}{x}}{1 - \frac{1}{\sqrt{x}}}$ still $\dfrac{0}{0}$

$= \lim_{x \to 1} \dfrac{\frac{1}{x^2}}{\frac{1}{2}x^{-3/2}} = \dfrac{1}{\frac{1}{2}} = 2,$ **C**

Name: ______________________________ Student #: ______________

7. *[3 marks]*
If $x_1 = 0$ is used as a first estimate to approximate a root of $x^3 + x = 1$ by Newton's method, then the third estimate, x_3, equals

A. $\frac{3}{4}$

B. $\frac{2}{3}$

C. $\frac{5}{6}$

D. $\frac{1}{2}$

E. $\frac{5}{8}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad f(x) = x^3 + x - 1$$
$$= x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} \qquad x_1 = 0$$
$$x_2 = 0 - \left(\frac{-1}{1}\right) = 1$$
$$x_3 = 1 - \frac{1+1-1}{4} = 1 - \frac{1}{4} = \frac{3}{4}, \qquad \textbf{A}$$

8. *[3 marks]*
$\int_1^e (\frac{1}{x} - \frac{1}{x^2})\, dx =$

A. $\frac{1}{e} - \frac{1}{e^2}$

B. $-\frac{1}{e^2} + \frac{2}{e^3}$

C. $\frac{1}{e} - 1$

D. $\frac{1}{e}$

E. 1

$$\left[\ln|x| + \frac{1}{x}\right]_1^e = (\ln e + \frac{1}{e}) - (\ln 1 + 1)$$
$$= 1 + \frac{1}{e} - 1 = \frac{1}{e}, \qquad \textbf{D}$$

Name: ____________________ Student #: ____________

9. *[3 marks]*
$$\int_0^2 x^2\sqrt{1+x^3}\,dx =$$

A. $\sqrt{35}$

B. $\sqrt{6}$

C. $\dfrac{26}{3}$

D. $\dfrac{52}{9}$

E. 36

Let $u = 1 + x^3$ $\qquad x = 0\ u = 1$
$du = 3x^2dx$ $\qquad x = 2\ u = 9$
$\dfrac{du}{3} = x^2dx$

$$\frac{1}{3}\int_1^9 \sqrt{u}\,du = \frac{1}{3} * \frac{2}{3}u^{3/2}\Big|_1^9$$
$$= \frac{2}{9}(27-1)$$
$$= \frac{52}{9}, \qquad \mathbf{D}$$

10. *[3 marks]*
Let $g(x) = \displaystyle\int_1^x \sqrt{2^t+1}\,dt$. Then $g'(2) =$

A. $\sqrt{5}$

B. $\sqrt{5}-\sqrt{2}$

C. $\dfrac{2\ln 2}{\sqrt{5}}$

D. $\sqrt{2^x+1}$

E. 0

$g'(x) = \sqrt{2^x+1}$
$g'(2) = \sqrt{2^2+1}$
$= \sqrt{5}$, **A**

Name: ____________________ Student #: ____________________

11. *[3 marks]*
Let $f(x,y) = x^2y^3 + e^{xy}$. Then $f_y = 3x^2y^2 + xe^{xy}$ **E**

A. $6xy^2 + xye^{xy}$

B. $3y^2 + e^x$

C. $3x^2y^2 + e^{xy}$

D. $3x^2y^2 + ye^{xy}$

E. $3x^2y^2 + xe^{xy}$

12. *[3 marks]*
If $y + xz^2 - x^2z^3 = 2$ defines z implicitly as a function of x and y near the point $x = 1$, $y = 2$, $z = 1$, then at that point $\dfrac{\partial z}{\partial y} =$

A. -1

B. 1

C. 0

D. -6

E. 5

$$1 + 2xz\frac{\partial z}{\partial y} - 3x^2z^2\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{1}{3x^2z^2 - 2xz} = 1 \text{ at } x = 1, y = 2, z = 1, \qquad \mathbf{B}$$

Name: ______________________ Student #: ______________

13. *[3 marks]*
If $f(x,y) = x^2 e^{y^2}$ then $f_{xy}(1,1) =$

A. $4e$

B. $12e$

C. $6e$

D. $8e$

E. $2e$

$f_x = 2xe^{y^2}$
$f_{xy} = 4xye^{y^2}$
$= 4e$ at $(1,1)$, **A**

14. *[3 marks]*
If $z = x^2 + xy + y^2$ where $x = 3t - 6$ and $y = t^2 + 2$ then when $t = 1$ $\dfrac{dz}{dt} =$

A. 9

B. 3

C. -3

D. 15

E. 6

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= (2x + y) * 3 + (x + 2y) * 2t$$
when $t = 1$ $x = -3$, $y = 3$
$$\frac{dz}{dt} = (-6 + 3) * 3 + (-3 + 6) * 2$$
$= -9 + 6 = -3$, **C**

15. *[3 marks]*
The joint demand functions for the products A and B are given by:

$$q_A = \frac{200}{p_A\sqrt{p_B}} \qquad\qquad q_B = \frac{300}{p_B\sqrt[3]{p_A}}$$

Which of the following statements is true?

A. $\dfrac{\partial q_A}{\partial p_A} > 0$

B. $\dfrac{\partial q_B}{\partial p_B} > 0$

C. Products A and B are complementary

D. Products A and B are competitive

E. Products A and B are neither complementary nor competitive

$\dfrac{\partial q_A}{\partial p_A} = \dfrac{-200}{p_A^2\sqrt{p_B}} < 0$ and $\dfrac{\partial q_B}{\partial p_B} = \dfrac{-300}{p_B\sqrt[3]{p_A}} < 0$

so A and B are false.

$\dfrac{\partial q_A}{\partial p_B} = \dfrac{-100}{p_A(p_B)^{3/2}} < 0$ and $\dfrac{\partial q_B}{\partial p_A} = \dfrac{-100}{p_B(p_A)^{4/3}} < 0$

so the goods are complementary. **C**

Name: ______________________________ Student #: ______________

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[11 marks]*

(a) *[4 marks]*
A 20 year mortgage for $500,000 has monthly payments with interest at 4% compounded semiannually. Find the amount of each payment (to the nearest cent).

If i is montly rate $(1+i)^{12} = (1.02)^2$

$500,000 = Ra_{\overline{240}|i}$

$$R = \frac{500,000}{a_{\overline{240}|i}} = \frac{500,000i}{1-(1+i)^{-240}} = \frac{500,000[(1.02)^{1/6}-1]}{1-(1.02)^{-40}}$$

$\boxed{\text{R=\$3021.23}}$

(b) *[3 marks]*
Find the principal outstanding (to the nearest cent) in the mortgage of question 1.(a), just after the 144th payment has been made.

Principal outstanding is the P.V. of the remaining $240 - 144 = 96$ payments

$$\text{P.O.} = Ra_{\overline{96}|i} = \frac{500,000}{a_{\overline{240}|i}} a_{\overline{96}|i} = 500,000\frac{\frac{1-(1+i)^{-96}}{i}}{\frac{(1-(1+i)^{-240})}{i}}$$

$$= 500,000\frac{[1-(1.02)^{-16}]}{[1-(1.02)^{-40}]} = \boxed{\$248{,}171.66}$$

(c) *[4 marks]*
What is the market price (to the nearest cent) of a $100 bond having 9 years until maturity and semiannual coupons, with annual coupon rate 6% and annual yield rate 5%?

$P = V(1+i)^{-n} + rVa_{\overline{n}|i} \qquad V = 100 \qquad n = 18 \qquad r = .03 \qquad i = .025$

$P = 100(1.025)^{-18} + 3a_{\overline{18}|.025}$

$= \boxed{\$107.18}$

Name: ______________________ Student #: ______________

B2. *[11 marks]*
Consider the graphs of $x = y^2 + 1$ and $x = 4y + 1$

(a) *[2 marks]*
Find the points where those graphs intersect.

$y^2 + 1 = 4y + 1$

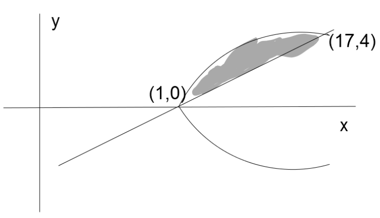
$y^2 - 4y = 0$

$y(y-4) = 0$

$y = 0 \Rightarrow x = 1 \qquad y = 4 \Rightarrow x = 17$

$\boxed{(1,0) \text{ and } (17,4)}$ are intersection points

(b) *[5 marks]*
Express as an integral the finite area bounded by those graphs.



Area$= \int_0^4 [(4y+1) - (y^2+1)]\, dy$

or

upper curve is $y = \sqrt{x-1}$ lower curve is $y = \dfrac{x-1}{4}$

Area$= \int_1^{17} \left[\sqrt{x-1} - \dfrac{x-1}{4}\right]\, dx$

(c) *[4 marks]*
Find the area from part (b).

First way: $\int_0^4 \left[(4y+1) - (y^2+1)\right]\, dy = \left[2y^2 - \dfrac{y^3}{3}\right]_0^4 = 2 * 16 - \dfrac{64}{3} - 0 = \boxed{\dfrac{32}{3}}$

Second way: $\int_1^{17} [\sqrt{x-1} - \dfrac{x-1}{4}]\, dx$

$$= \left[\frac{2}{3}(x-1)^{3/2} - \frac{(x-1)^2}{8}\right]_1^{17}$$

$$= \frac{2}{3} * 16^{3/2} - \frac{1}{8} * 16^2 - 0$$

$= \dfrac{2}{3} * 64 - 32 = \boxed{\dfrac{32}{3}}$ as well

Name: ____________________ Student #: ____________________

B3. *[13 marks]*

Evaluate the following integrals

(a) *[6 marks]*

$$\int_0^1 (2x+1)e^{2x}\,dx$$

By parts $u = 2x+1 \qquad du = 2dx$

$$dv = e^{2x} \qquad v = \frac{e^{2x}}{2}$$

$$\begin{aligned}\int_0^1 (2x+1)e^{2x}\,dx &= \left.\frac{(2x+1)e^{2x}}{2}\right|_0^1 - \int_0^1 e^{2x}\,dx \\ &= \frac{3e^2}{2} - \frac{1}{2} - \left.\frac{e^{2x}}{2}\right|_0^1 \\ &= \frac{3e^2}{2} - \frac{1}{2} - \left(\frac{e^2}{2} - \frac{1}{2}\right) = \boxed{e^2}\end{aligned}$$

(b) *[7 marks]*

$$\int_1^\infty \frac{1}{x^2(x+1)}\,dx = \lim_{R\to\infty}\int_1^R \frac{dx}{x^2(x+1)}$$

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \qquad Ax(x+1) + B(x+1) + Cx = 1$$

$$x = \quad 0 \Rightarrow B = 1$$

$$x = -1 \Rightarrow C = 1$$

$$x = \quad 1 \Rightarrow 2A + 2B + C = 1$$

$$2A + 2 \quad + 1 = 1 \Rightarrow A = -1$$

$$\begin{aligned}&\lim_{R\to\infty}\int_1^R \left[\frac{-1}{x} + \frac{1}{x+1} + \frac{1}{x^2}\right]\,dx \\ &= \lim_{R\to\infty}\left[-\ln|x| + \ln|x+1| - \frac{1}{x}\right]_1^R \\ &= \lim_{R\to\infty}\left[\left(\ln\frac{R+1}{R} - \frac{1}{R}\right) - (\ln 2 - 1)\right] \qquad \frac{R+1}{R} \to 1 \text{ so } \ln\left(\frac{R+1}{R}\right) \to 0 \text{ and } \frac{1}{R} \to 0 \\ &= \boxed{1 - \ln 2}\end{aligned}$$

Name: ______________________________ Student #: ______________

B4. *[8 marks]*

Assuming that $y > 0$, find an expression for y in terms of x if y satisfies the differential equation

$$\frac{dy}{dx} = xy$$

and $y = 3$ when $x = 0$. [Hint: This one's easy.]

$$\frac{dy}{y} = xdx$$

$$\int \frac{dy}{y} = \int x \, dx$$

$y > 0$ $\quad \ln y = \frac{x^2}{2} + C$

$$y = e^{x^2/2+C} = e^{x^2/2}e^C = Ae^{x^2/2}$$

when $x = 0$ $\quad 3 = Ae^0 = A$

so $\boxed{y = 3e^{x^2/2}}$

Name: ______________________________ Student #: ____________________

B5. *[12 marks]*

(a) *[6 marks]*
Find and classify the critical point(s) of $f(x, y) = 5x^2 - 2xy + 2y^2 - 10x + 2y$

$$\begin{aligned} f_x &= 10x - 2y - 10 = 0 \\ f_y &= -2x + 4y + 2 = 0 \end{aligned} \qquad \begin{aligned} 5x - y &= 5 \\ -x + 2y &= -1 \\ x - 2y &= 1 \\ 5x - y &= 5 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 5 & -1 & 5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 9 & 0 \end{array}\right)$$

$\boxed{y = 0 \text{ and } x = 1}$

$f_{xx} = 10 \quad f_{yy} = 4 \quad f_{xy} = -2$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$10 * 4 - (-2)^2 = 36 > 0$

$(1, 0)$ is a local extremum

$f_{xx} > 0$ so $\boxed{(1,0) \text{ is a local min}}$

(b) *[6 marks]*
By using the method of Lagrange multipliers **only** find the critical points of the joint cost function

$$c(q_A, q_B) = q_A^2 - q_A q_B + \frac{3}{2} q_B^2 + 300$$

subject to the constraint

$$q_A + q_B = 700.$$

[Show all your work. No marks will be given for any other method.]

$$\mathscr{L} = q_A^2 - q_A q_B + \frac{3}{2} q_B^2 + 300 - \lambda(q_A + q_B - 700)$$

$$\frac{\partial \mathscr{L}}{\partial q_A} = 2q_A - q_B - \lambda = 0 \qquad \text{and} \qquad \frac{\partial \mathscr{L}}{\partial \lambda} = -(q_A + q_B - 700) = 0$$

$$\frac{\partial \mathscr{L}}{\partial q_B} = -q_A + 3q_B - \lambda = 0 \qquad \text{so } q_A + q_B = 700$$

$2q_A - q_B = \lambda$

$-q_A + 3q_B = \lambda$

$2q_A - q_B = -q_A + 3q_B$

$$\left.\begin{aligned} 3q_A - 4q_B &= 0 \\ q_A + q_B &= 700 \end{aligned}\right\} \qquad \left(\begin{array}{cc|c} 1 & 1 & 700 \\ 3 & -4 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 700 \\ 0 & -7 & -2100 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 700 \\ 0 & 1 & 300 \end{array}\right) \qquad \boxed{q_B = 300 \quad q_A = 400}$$