FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL 2013

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

Duration: 3 hours
Examiners: A. Igelfeld
P. Kergin
J. Tate
P. Walls

FAMILY NAME: ____________________

GIVEN NAME: ____________________

STUDENT NO: ____________________

SIGNATURE: ____________________

LEAVE BLANK	
Question	Mark
MC	/45
B1	/11
B2	/11
B3	/10
B4	/10
B5	/13
TOTAL	

NOTE:

1. **Aids Allowed:** A non-graphing calculator, with empty memory, to be supplied by student.
2. **Instructions:** Fill in the information on this page, and make sure your test booklet contains 14 pages.
3. This exam consists of 15 multiple choice questions, and 5 written-answer questions. For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the front page** with your pencil. Each correct answer is worth 3 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.
4. Put your name and student number on each page of this examination.

ANSWER BOX FOR PART A					
Circle the correct answer					
1.	**A.**	**B.**	**C.**	**D.**	**E.**
2.	**A.**	**B.**	**C.**	**D.**	**E.**
3.	**A.**	**B.**	**C.**	**D.**	**E.**
4.	**A.**	**B.**	**C.**	**D.**	**E.**
5.	**A.**	**B.**	**C.**	**D.**	**E.**
6.	**A.**	**B.**	**C.**	**D.**	**E.**
7.	**A.**	**B.**	**C.**	**D.**	**E.**
8.	**A.**	**B.**	**C.**	**D.**	**E.**
9.	**A.**	**B.**	**C.**	**D.**	**E.**
10.	**A.**	**B.**	**C.**	**D.**	**E.**
11.	**A.**	**B.**	**C.**	**D.**	**E.**
12.	**A.**	**B.**	**C.**	**D.**	**E.**
13.	**A.**	**B.**	**C.**	**D.**	**E.**
14.	**A.**	**B.**	**C.**	**D.**	**E.**
15.	**A.**	**B.**	**C.**	**D.**	**E.**

NAME: ______________________________ STUDENT #: __________________

Record your answers on the front page

PART A. MULTIPLE CHOICE

1. *[3 marks]*

The slope of the tangent to the curve $f(x) = \dfrac{(x^2-3)^3}{\sqrt{4-3x}}$ at the point where $x = 1$ is:

A. -16

B. -12

C. $25\,\dfrac{1}{2}$

D. 12

E. 16

2. *[3 marks]*

Given: $y = \dfrac{3^{x^2} e^{-x}}{(x+2)^{3x}}$, find $\dfrac{dy}{dx}$ when $x = -1$.

A. $6e(1 - \ln 3)$

B. $3e$

C. $2 - 2\ln 3$

D. $-2e\ln 3$

E. undefined

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Record your answers on the front page

3. *[3 marks]*

A horizontal asymptote for the graph of $y = e^{-1/x^2}$ is

A. $x = 0$

B. $x = 1$

C. $y = 0$

D. $y = 1$

E. $y = x$

4. *[3 marks]*

Let $f(x) = (x^2 - 1)^2$ on the interval given by $-1 < x < 2$. Then f has

A. an absolute minimum at $x = 1$ and also at $x = -1$

B. an absolute minimum at $x = 1$ and no absolute maximum

C. neither an absolute minimum nor an absolute maximum

D. an absolute minimum at $x = 1$ and an absolute maximum at $x = 2$

E. an absolute minimum at $x = 1$ and an absolute maximum at $x = 0$

5. *[3 marks]*

$\lim_{x \to 0^+} (x+1)^{\left(1/x^2\right)} =$

A. 1

B. $e + 1$

C. 0

D. ∞

E. e

6. *[3 marks]*

$\int_{-1}^{1} (x^2+1)(1-x^2)dx =$

A. 0

B. $\frac{2}{5}$

C. $\frac{8}{5}$

D. 2

E. $-\frac{2}{5}$

7. *[3 marks]*

If $f''(x) = e^x - 1$, $f'(0) = -1$, and $f(0) = 1$, then $f(x) =$

A. $e^x - \frac{1}{2}x^2 - 2x$

B. $e^x - \frac{1}{2}x^2 - x + 1$

C. $e^x - x^2 - 2x$

D. $e^x - \frac{1}{2}x^2$

E. $e^x - x^2 + x$

8. *[3 marks]*

The average value of $f(x) = xe^{-x^2}$ over the interval $[0, 2]$ is

A. $\dfrac{1 - e^{-4}}{4}$

B. 1

C. $\dfrac{1 - e^{-2}}{2}$

D. $1 - e^{-4}$

E. $\dfrac{e^{-2} - 1}{4}$

9. *[3 marks]*

If $F'(x) = f(x)$, $F(2) = 3$, and $F(0) = -2$, then $\int_0^2 f(x)dx =$

A. -5

B. 5

C. 2

D. -2

E. 1

10. *[3 marks]*

If $f(x) = \int_e^x \frac{dt}{\ln t}$, then $f'(e^3) =$

A. $\frac{1}{3}$

B. $-\frac{1}{3e^3}$

C. $-\frac{1}{9e^3}$

D. $3e^2$

E. 1

11. *[3 marks]*

If a certain good has demand function $p = 15 - q^2$ and supply function $p = 2q$, what is the consumers' surplus for the good? Note: this good has equilibrium point $q = 3, \quad p = 6$.

A. 27

B. 24

C. 18

D. 12

E. 30

12. *[3 marks]*

If $f(x, y) = 3x^2 - x^3y^2 + xy^4 - 4x$, $f_{xyy}(1, 1) =$

A. 6

B. 0

C. -1

D. 2

E. 4

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Record your answers on the front page

13. *[3 marks]*

Given $z^3 + x^2y - y^2x - xz = 0$, then at $x = y = z = 1$, $\dfrac{\partial z}{\partial x} =$

A. 4

B. 2

C. -1

D. 0

E. undefined

14. *[3 marks]*

If $z = x^2e^y + y^2e^x$ where $x = 2rs^2$ and $y = 2\ln r + 3\ln s$, then when $r = 1$ and $s = 1$, $\dfrac{\partial z}{\partial s} =$

A. 0

B. 1

C. $5e$

D. $2e$

E. 28

15. *[3 marks]*

If $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix}$, and $AX = B$, then $X =$

A. $\begin{bmatrix} -1 & -3 \\ 0 & 4 \end{bmatrix}$

B. $\begin{bmatrix} -2 & -8 \\ 0 & 4 \end{bmatrix}$

C. $\begin{bmatrix} -1 & -4 \\ 0 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

E. $\begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$

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PART B. WRITTEN-ANSWER QUESTIONS

B1. *[11 marks]*

Parts (a) and (b) below concern a 20 year mortgage for $300,000 with monthly payments and interest at 4% compounded semiannually. In parts (a), (b), and (c), give answers to the nearest cent.

[4 marks] (a) Find the amount of each payment of the mortgage.

[4 marks] (b) Just after the first 15 years of the mortgage, what is the outstanding principal?

[3 marks] (c) If a $100 bond has 9 years to maturity with semiannual coupons at annual coupon rate 4.4% and annual yield rate 4%, what is its market price?

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B2. *[11 marks]*

[1 mark] (a) Find the points of intersection of the the curves $y = x\,2^x$ and $y = 2x$.

[10 marks] (b) Find the area bounded by the two curves.

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B3. *[10 marks]*

Find $\displaystyle\int_0^\infty \frac{1}{x^2 + 3x + 2} dx$ or show that the integral diverges.

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B4. *[10 marks]*

Find the unique solution to the differential equation $xy' = y^2$ satisfying $y(e) = -\frac{1}{2}$. What is the value of y when $x = e^2$?

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B5. *[13 marks]*

[6 marks] (a) Let $f(x, y) = x^2 + 6xy + 3y^2 - 16x - 36y$. Find any critical point(s) of f and classify each one.

[7 marks] (b) If the Acme company charges p dollars per unit for its product, it will sell q units, provided $3p + 2q = 60$. Use **Lagrange multipliers** to find the values of p and q which maximize the Acme company's revenue.

Note: **No** marks will be assigned to any alternate method of solution. You don't need to verify that your solution actually maximizes revenue, but you do need to find the value of the Lagrange multiplier.

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SOLUTIONS

PART A. MULTIPLE CHOICE

1. *[3 marks]*

The slope of the tangent to the curve $f(x) = \dfrac{(x^2-3)^3}{\sqrt{4-3x}}$ at the point where $x = 1$ is:

A. -16

B. -12

C. $25\,\dfrac{1}{2}$

D. 12

E. 16

$y' = \dfrac{\sqrt{4-3x}\cdot 3(x^2-3)^2\cdot 2x - (x^2-3)^3\cdot\frac{1}{2\sqrt{4-3x}}\cdot(-3)}{4-3x}.$

At $x=1$, $y' = 3(-2)^2\cdot 2 - (-2)^3(-\frac{3}{2}) = 24 - 12 = 12,\quad$ **D**.

2. *[3 marks]*

Given: $y = \dfrac{3^{x^2}e^{-x}}{(x+2)^{3x}}$, find $\dfrac{dy}{dx}$ when $x = -1$.

A. $6e(1-\ln 3)$

B. $3e$

C. $2 - 2\ln 3$

D. $-2e\ln 3$

E. undefined

$\ln y = x^2\ln 3 - x - 3x\ln(x+2)$

$\frac{1}{y}y' = 2x\ln 3 - 1 - 3\ln(x+2) - \frac{3x}{x+2}$

At $x = 1, y = 3e$

$\frac{1}{3e}y' = -2\ln 3 - 1 - 3\ln 1 + 3 = -2\ln 3 + 2$

$y' = 6e(-\ln 3 + 1),\quad$ **A**.

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SOLUTIONS

3. *[3 marks]*

A horizontal asymptote for the graph of $y = e^{-1/x^2}$ is

A. $x = 0$

B. $x = 1$

C. $y = 0$

D. $y = 1$

E. $y = x$

$$\lim_{x \to \pm\infty} \frac{1}{x^2} = 0$$
$$\lim_{x \to \pm\infty} e^{-\frac{1}{x^2}} = e^0 = 1, \quad \textbf{D}.$$

4. *[3 marks]*

Let $f(x) = (x^2 - 1)^2$ on the interval given by $-1 < x < 2$. Then f has

A. an absolute minimum at $x = 1$ and also at $x = -1$

B. an absolute minimum at $x = 1$ and no absolute maximum

C. neither an absolute minimum nor an absolute maximum

D. an absolute minimum at $x = 1$ and an absolute maximum at $x = 2$

E. an absolute minimum at $x = 1$ and an absolute maximum at $x = 0$

$f'(x) = 2(x^2 - 1) \cdot 2x = 0$ at $x = 0$ and $x = 1$.

	f'	f
$(-1, 0)$	$+$	inc
$(0, 1)$	$-$	dec
$(1, 2)$	$+$	inc

$x = 0$ is a local max, $x = 1$ is a local min.

Since $f(x) \geq 0$ always, the local min at $x = 1$ *is a global minimum.* $x = 0$ is a local max, but since near $x = 2$, $f(x)$ is near 9 and $f(0)$ is only 1, there is no global max. **B**.

Note: $x = -1$ is not a point where f is defined; neither is $x = 2$.

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SOLUTIONS

5. *[3 marks]*

$\lim_{x\to 0^+} (x+1)^{\left(1/x^2\right)} =$

A. 1

B. $e+1$

C. 0

D. ∞

E. e

Let $y = (x+1)^{\frac{1}{x^2}}$

$\ln y = \dfrac{\ln(x+1)}{x^2}$ is of the form $\left(\frac{0}{0}\right)$ near 0.

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \dfrac{1}{2x(x+1)} \to \infty$ since $x+1 \to 1$ and $x \to 0^+$, **D**

6. *[3 marks]*

$\int_{-1}^{1} (x^2+1)(1-x^2)dx =$

A. 0

B. $\frac{2}{5}$

C. $\frac{8}{5}$

D. 2

E. $-\frac{2}{5}$

$\int_{-1}^{1} (1-x^4)dx = \left(x - \dfrac{x^5}{5}\right)\Big|_{-1}^{1}$

$= \left(1 - \dfrac{1}{5}\right) - \left(-1 - \dfrac{(-1)^5}{5}\right)$

$= \dfrac{4}{5} - \left(-\dfrac{4}{5}\right) = \dfrac{8}{5}$, **C**

Or, by symmetry:

$= 2\int_{0}^{1} (1-x^4)dx = 2\left(x - \dfrac{x^5}{5}\right)\Big|_{0}^{1} = \dfrac{8}{5}$, **C**

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SOLUTIONS

7. *[3 marks]*

If $f''(x) = e^x - 1, \;\; f'(0) = -1, \;$ and $f(0) = 1, \;$ then $f(x) =$

A. $e^x - \frac{1}{2}x^2 - 2x$

B. $e^x - \frac{1}{2}x^2 - x + 1$

C. $e^x - x^2 - 2x$

D. $e^x - \frac{1}{2}x^2$

E. $e^x - x^2 + x$

$f'(x) = e^x - x + C$
$-1 = f'(0) = 1 + C$, so $C = -2$.
$f'(x) = e^{-x} - x - 2$
$f(x) = e^x - \frac{x^2}{2} - 2x + C$
$1 = f(0) = 1 + C$ so $C = 0$.
$f(x) = e^x - \frac{x^2}{2} - 2x,$ **A**

8. *[3 marks]*

The average value of $f(x) = xe^{-x^2}$ over the interval $[0, 2]$ is

A. $\dfrac{1 - e^{-4}}{4}$

B. 1

C. $\dfrac{1 - e^{-2}}{2}$

D. $1 - e^{-4}$

E. $\dfrac{e^{-2} - 1}{4}$

$$\frac{1}{2}\int_0^2 xe^{-x^2}\,dx$$
Let $u = x^2 \implies du = 2xdx \implies \frac{1}{2}du = xdx$
$$= \frac{1}{4}\int_0^4 e^{-u}du$$
$$= \tfrac{1}{4}(-e^{-u})\Big|_0^4$$
$$= \frac{1}{4}\left(-e^{-4} - (-1)\right) = \frac{1 - e^{-4}}{4},$$ **A**

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SOLUTIONS

9. *[3 marks]*

If $F'(x) = f(x)$, $F(2) = 3$, and $F(0) = -2$, then $\int_0^2 f(x)dx =$

A. -5

B. 5

C. 2

D. -2

E. 1

$$\int_0^2 f(x)dx = \int_0^2 F'(x)dx = F(2) - F(0) = 3 - (-2) = 5, \quad \mathbf{B}.$$

10. *[3 marks]*

If $f(x) = \int_e^x \frac{dt}{\ln t}$, then $f'(e^3) =$

A. $\frac{1}{3}$

B. $-\frac{1}{3e^3}$

C. $-\frac{1}{9e^3}$

D. $3e^2$

E. 1

$f'(x) = \frac{1}{\ln x}$

$f'(e^3) = \frac{1}{\ln e^3} = \frac{1}{3}$, **A**.

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SOLUTIONS

11. *[3 marks]*

If a certain good has demand function $p = 15 - q^2$ and supply function $p = 2q$, what is the consumers' surplus for the good? Note: this good has equilibrium point $q = 3, \;\; p = 6$.

A. 27

B. 24

C. 18

D. 12

E. 30

$CS = \int_0^{q_0} [D(q) - p_0] dq = \int_0^3 [(15 - q^2) - 6] dq = \int_0^3 (9 - q^3) dq = (9q - \frac{q^3}{3})\Big|_0^3 = 27 - 9 = 18$, **C**.

12. *[3 marks]*

If $f(x, y) = 3x^2 - x^3y^2 + xy^4 - 4x$, $f_{xyy}(1, 1) =$

A. 6

B. 0

C. -1

D. 2

E. 4

Using the equality of mixed partials to differentiate by y first,
$f_y = -2x^3y + 4xy^3$
$f_{yy} = -2x^3 + 12xy^2$
$f_{yyx} = -6x^2 + 12y^2 = 6$ at $(1, 1)$, **A**.

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SOLUTIONS

13. *[3 marks]*

Given $z^3 + x^2y - y^2x - xz = 0$, then at $x = y = z = 1$, $\dfrac{\partial z}{\partial x} =$

A. 4

B. 2

C. -1

D. 0

E. undefined

$3z^2\frac{\partial z}{\partial x} + 2xy - y^2 - z - x\frac{\partial z}{\partial x} = 0$
At $x = y = z = 1$,
$3\frac{\partial z}{\partial x} + 2 - 1 - 1 - \frac{\partial z}{\partial x} = 0$
$2\frac{\partial z}{\partial x} = 0$
$\frac{\partial z}{\partial x} = 0$, **D**

14. *[3 marks]*

If $z = x^2e^y + y^2e^x$ where $x = 2rs^2$ and $y = 2\ln r + 3\ln s$, then when $r = 1$ and $s = 1$, $\dfrac{\partial z}{\partial s} =$

A. 0

B. 1

C. $5e$

D. $2e$

E. 28

$\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$
$= (2xe^y + y^2e^x)4rs + (x^2e^y + 2ye^x)\dfrac{3}{s}$
At $r = s = 1, x = 2$ and $y = 0$, so $\dfrac{\partial z}{\partial s} = 4 \cdot 4 + 4 \cdot 3 = 28$, **E**.

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SOLUTIONS

15. *[3 marks]*

If $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix}$, and $AX = B$, then $X =$

A. $\begin{bmatrix} -1 & -3 \\ 0 & 4 \end{bmatrix}$

B. $\begin{bmatrix} -2 & -8 \\ 0 & 4 \end{bmatrix}$

C. $\begin{bmatrix} -1 & -4 \\ 0 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

E. $\begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$

$X = A^{-1}B$. To find A^{-1} :

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{C}$$

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PART B. WRITTEN-ANSWER QUESTIONS

B1. *[11 marks]*

Parts (a) and (b) below concern a 20 year mortgage for \$300,000 with monthly payments and interest at 4% compounded semiannually. In parts (a), (b), and (c), give answers to the nearest cent.

[4 marks] (a) Find the amount of each payment of the mortgage.

$$300,000 = Ra_{\overline{240}|i}$$

$$R = \frac{300,000i}{1-(1+i)^{-240}}$$

$$(1+i)^{12} = (1.02)^2$$

$$(1+i)^{-240} = (1.02)^{-40}$$

$$R = \frac{300,000((1.02)^{1/6}-1)}{1-(1.02)^{-40}} \approx \boxed{\$1812.74}$$

[4 marks] (b) Just after the first 15 years of the mortgage, what is the outstanding principal?

5 years remain.

$$P.O. = Ra_{\overline{60}|i} = 1812.74\frac{(1-(1+i)^{-60})}{i}$$

$$= 1812.74\frac{(1-(1.02)^{-10})}{(1.02)^{1/6}-1}$$

$$\approx \boxed{\$98{,}509.57.}$$

Easier is:

$$P.O. = Ra_{\overline{60}|i} = 300,000\frac{a_{\overline{60}|i}}{a_{\overline{240}|i}} = 300,000\frac{1-(1.02)^{-10}}{1-(1.02)^{-40}} \approx \boxed{\$98{,}509.53}$$

[3 marks] (c) If a \$100 bond has 9 years to maturity with semiannual coupons at annual coupon rate 4.4% and annual yield rate 4%, what is its market price?

$$P = V(i+1)^{-n} + rVa_{\overline{n}|i}$$

$$n = 18, V = 100$$

$$r = 0.022, i = 0.02$$

$$P = 100(1.02)^{-18} + 2.20a_{\overline{18}|.02}$$

$$= 102.9984\cdots \approx \boxed{\$103}$$

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B2. *[11 marks]*

[1 mark] (a) Find the points of intersection of the the curves $y = x\,2^x$ and $y = 2x$.

$$\begin{aligned} x2^x &= 2x \\ x2^x - 2x &= 0 \\ x(2^x - 2) &= 0, \end{aligned}$$

so $\boxed{x = 0, y = 0}$ and $\boxed{x = 1, y = 2}$ are the points of intersection.

[10 marks] (b) Find the area bounded by the two curves.

The interval is $[0, 1]$. At $x = \frac{1}{2}$, $x2^x = \frac{\sqrt{2}}{2}$ and $2x = 1$. Since $\frac{\sqrt{2}}{2} < 1$, $y = 2x$ lies above $y = x2^x$ on $[0, 1]$ and the area is:

$$A = \int_0^1 (2x - x2^x)dx = x^2\Big|_0^1 - \int_0^1 x2^x dx.$$

Let $u = x, dv = 2^x dx \implies du = dx, v = \frac{2^x}{\ln 2}$.

$$\begin{aligned} A &= 1 - \frac{x2^x}{\ln 2}\Big|_0^1 + \frac{1}{\ln 2}\int_0^1 2^x dx \\ &= 1 - \frac{2}{\ln 2} + \frac{2^x}{(\ln 2)^2}\Big|_0^1 \\ &= 1 - \frac{2}{\ln 2} + \frac{2}{(\ln 2)^2} - \frac{1}{(\ln 2)^2} \\ &= \boxed{1 - \frac{2}{\ln 2} + \frac{1}{(\ln 2)^2}.} \end{aligned}$$

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B3. *[10 marks]*

Find $\displaystyle\int_0^\infty \frac{1}{x^2+3x+2}dx$ or show that the integral diverges.

$= \displaystyle\lim_{R\to\infty}\int_0^R \frac{1}{(x+2)(x+2)}dx$

$\displaystyle\frac{1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1} \implies A(x+1)+B(x+2) = 1$. Plugging in $x = -1$, we get $B = 1$, and from $x = -2$, we get $-A = 1$.

$$\begin{aligned}\lim_{R\to\infty}\int_0^R \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx &= \lim_{R\to\infty} (\ln|x+1| - \ln|x+2|)\Big|_0^R \\ &= \lim_{R\to\infty} \ln\left(\left|\frac{R+1}{R+2}\right|\right) - \ln\left(\frac{1}{2}\right) \\ &= \ln 1 + \ln 2 = \boxed{\ln 2}\end{aligned}$$

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B4. *[10 marks]*

Find the unique solution to the differential equation $xy' = y^2$ satisfying $y(e) = -\frac{1}{2}$. What is the value of y when $x = e^2$?

$$\begin{aligned} x\frac{dy}{dx} &= y^2 \\ \frac{dy}{y^2} &= \frac{dx}{x} \\ \int \frac{dy}{y^2} &= \int \frac{dx}{x} \\ \frac{-1}{y} &= \ln|x| + C \end{aligned}$$

At $x = e, y = -\frac{1}{2}$, so $2 = 1 + C$, and $C = 1$.

$$\begin{aligned} -\frac{1}{y} &= \ln|x| + 1 \\ y &= -\frac{1}{1 + \ln|x|}. \end{aligned}$$

When $x = e^2, y = -\frac{1}{1+2} = \boxed{-\frac{1}{3}}$.

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B5. *[13 marks]*

[6 marks] (a) Let $f(x, y) = x^2 + 6xy + 3y^2 - 16x - 36y$. Find any critical point(s) of f and classify each one.

$$\begin{aligned} f_x &= 2x + 6y - 16 = 0 \implies x + 3y = 8 \\ f_y &= 6x + 6y - 36 = 0 \implies x + y = 6 \end{aligned}$$

so $2y = 2$, so $y = 1, x = 5$. The only critical point is $\boxed{x = 5, y = 1}$.

$$\begin{aligned} f_{xx} &= 2, f_{yy} = 6, f_{xy} = 6. \\ D &= f_{xx}f_{yy} - (f_{xy})^2 = 2 \cdot 6 - 6^2 = -24 < 0, \end{aligned}$$

so $\boxed{\text{(5,1) is a saddle point.}}$

[7 marks] (b) If the Acme company charges p dollars per unit for its product, it will sell q units, provided $3p + 2q = 60$. Use **Lagrange multipliers** to find the values of p and q which maximize the Acme company's revenue.

Note: **No** marks will be assigned to any alternate method of solution. You don't need to verify that your solution actually maximizes revenue, but you do need to find the value of the Lagrange multiplier.

$R = pq$ subject to $3p + 2q = 60$.

$$\begin{aligned} \mathcal{L} &= pq - \lambda(3p + 2q - 60) \\ \mathcal{L}_p &= q - 3\lambda = 0 \implies q = 3\lambda \\ \mathcal{L}_q &= p - 2\lambda = 0 \implies p = 2\lambda \\ \mathcal{L}_\lambda &= -(3p + 2q - 60) = 0 \\ 60 &= 3p + 2q = 6\lambda + 6\lambda = 12\lambda, \end{aligned}$$

so $\boxed{\lambda = 5, p = 10, q = 15.}$