FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL 2012

MAT 133Y1Y
Calculus and Linear Algebra for Commerce

Duration: 3 hours
Examiners: N. Francetic
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P. Kergin
J. Tate

LEAVE BLANK	
Question	Mark
MC	/45
B1	/13
B2	/12
B3	/10
B4	/10
B5	/10
TOTAL	

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

NOTE:

1. **Aids Allowed:** A non-graphing calculator, with empty memory, to be supplied by student.
2. **Instructions:** Fill in the information on this page, and make sure your test booklet contains 14 pages.
3. This exam consists of 15 multiple choice questions, and 5 written-answer questions. For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the front page** with your pencil. Each correct answer is worth 3 marks; a question left blank, or an incorrect answer or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.
4. Put your name and student number on each page of this examination.

ANSWER BOX FOR PART A					
Circle the correct answer					
1.	**A.**	**B.**	**C.**	**D.**	**E.**
2.	**A.**	**B.**	**C.**	**D.**	**E.**
3.	**A.**	**B.**	**C.**	**D.**	**E.**
4.	**A.**	**B.**	**C.**	**D.**	**E.**
5.	**A.**	**B.**	**C.**	**D.**	**E.**
6.	**A.**	**B.**	**C.**	**D.**	**E.**
7.	**A.**	**B.**	**C.**	**D.**	**E.**
8.	**A.**	**B.**	**C.**	**D.**	**E.**
9.	**A.**	**B.**	**C.**	**D.**	**E.**
10.	**A.**	**B.**	**C.**	**D.**	**E.**
11.	**A.**	**B.**	**C.**	**D.**	**E.**
12.	**A.**	**B.**	**C.**	**D.**	**E.**
13.	**A.**	**B.**	**C.**	**D.**	**E.**
14.	**A.**	**B.**	**C.**	**D.**	**E.**
15.	**A.**	**B.**	**C.**	**D.**	**E.**

Name: ______________________ Student #: ______________

Record your answers on the front page

PART A. MULTIPLE CHOICE

1. *[3 marks]*
If $PQ = R$

where $P = \begin{pmatrix} 1 & 5 & 7 & 8 & 9 \\ 3 & -1 & 0 & 4 & 2 \\ 2 & 4 & -2 & 1 & 6 \end{pmatrix}$

and $Q = \begin{pmatrix} 2 & 1 & 6 & 0 \\ 7 & -2 & 2 & -1 \\ 9 & 2 & 4 & 4 \\ 4 & 0 & 3 & 0 \\ 0 & 1 & 1 & 7 \end{pmatrix}$

then r_{24} is

A. 14
B. 15
C. 16
D. 17
E. undefined because P and Q cannot be multiplied together

2. *[3 marks]*
If $f(x) = \begin{cases} x + e^k & \text{if } x \leq 1 \\ \dfrac{2x^2 - 2x}{x - 1} & \text{if } x > 1 \end{cases}$
is continuous at $x = 1$ then

A. $k = -1$
B. $k = 0$
C. $k = 1$
D. $k = 2$
E. there is no real number k such that f is continuous at $x = 1$

3. *[3 marks]*
If $y = x^{\ln x}$, then $y'(e) =$

   A. $\dfrac{2}{e}$

   B. $\dfrac{1}{e}$

   C. e

   D. 2

   E. $\dfrac{1}{2e}$

4. *[3 marks]*
The function $f(x) = xe^{-x}$ on the inverval $[-1, 2]$

   A. has an absolute minimum at $x = -1$ and an absolute maximum at $x = 2$
   B. has an absolute minimum at $x = -1$ and an absolute maximum at $x = 1$
   C. has an absolute minimum at $x = 1$ and an absolute maximum at $x = 2$
   D. has an absolute maximum at $x = 1$ and no absolute minimum
   E. has an absolute minimum at $x = -1$ and no absolute maximum

Record your answers on the front page

5. *[3 marks]*
If the demand equation is $p = \dfrac{q^3}{12} - 50q^2 + 10000q$, for what values of q is the **marginal revenue** $(\dfrac{dr}{dq})$ decreasing?
   A. $(0, 100)$
   B. $(100, 200)$
   C. $(200, \infty)$
   D. All values
   E. None

6. *[3 marks]*
$\lim\limits_{x \to 0} \dfrac{e^{2x} - 1 - 2x - 2x^2}{x^3}$ is
   A. undefined
   B. 0
   C. 2
   D. $\dfrac{1}{6}$
   E. $\dfrac{4}{3}$

Record your answers on the front page

7. *[3 marks]*
If Newton's method is used to estimate a solution to the equation

$$(x-2)^2 = \ln x$$

beginning with $x_1 = 1$, then x_2 is closest to

A. 0.67

B. 1.33

C. 1.39

D. 1.41

E. 1.43

8. *[3 marks]*

$$\int_1^e \frac{\ln x}{x} dx =$$

A. 2

B. 1

C. $\frac{1}{2}$

D. e

E. $\frac{1}{e}$

9. *[3 marks]*

$$\int \frac{x+2}{x^2-1}dx =$$

A. $(x+2)\ln|x^2-1|+C$

B. $\frac{5}{2}\ln|x^2-1|+C$

C. $\frac{1}{2}[\ln|x+2|-\ln|x+1|-\ln|x-1|]+C$

D. $\frac{1}{2}[\ln|x-1|-\ln|x+1|]+C$

E. $\frac{1}{2}[3\ln|x-1|-\ln|x+1|]+C$

10. *[3 marks]*

A town's population is expected to grow exponentially so that it will double every 50 years. If its population today is 10,000, then in 10 years its population will be

A. $12,039$

B. $11,755$

C. $10,821$

D. $11,487$

E. $10,432$

11. *[3 marks]*

If $z = (x^2 + y^2)\ln(3x + y)$, then when $x = 0$ and $y = e$, $\dfrac{\partial z}{\partial x} =$

A. 0

B. 1

C. e^2

D. e

E. $3e$

12. *[3 marks]*

If $y^2 + z^2 = xyz + 1$, then when $x = y = z = 1$, $\dfrac{\partial z}{\partial y} =$

A. -1

B. 0

C. 1

D. 2

E. -2

13. *[3 marks]*

If $x = 3r + 5s$, $y = 2r + 3s$, and $z = x^2 + y^2$, then when $r = s = 1$, $\dfrac{\partial z}{\partial r} =$

- **A**. 64
- **B**. 10
- **C**. 42
- **D**. 68
- **E**. 39

14. *[3 marks]*

The quantities q_A and q_B of goods A and B, which are sold when the respective unit prices of the goods are p_A and p_B, are

$q_A(p_A, p_B) = 200 - 5p_A + 10p_B - p_B^2$
$q_B(p_A, p_B) = 100 - 6p_A - 3p_B + p_A^2$

The two goods are complementary

- **A**. when $p_A > 3$ and $p_B < 5$
- **B**. when $p_A > 3$ and $p_B > 5$
- **C**. when $p_A < 3$ and $p_B > 5$
- **D**. for no values of p_A and p_B
- **E**. when $p_A < 3$ and $p_B < 5$

15. *[3 marks]*
The critical point of
$$f(x,y) = x^2 + 2xy - y^2 - 2x - 6y + 3$$
is

A. $(2,-1)$, which is a saddle point
B. $(2,-1)$, which is a local max
C. $(2,-1)$, which is a local min
D. $(2,-1)$, but the second derivative test cannot determine its nature
E. not defined, because the partial derivatives are never both zero

Name: ______________________ Student #: ______________

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[13 marks]*
At the end of December, 2000, Henry invested \$100,000 in an annuity which could be refunded at any time and which made 20 equal semiannual payments at the ends of June and December from 2001 to 2010 inclusive at 6% compounded semi-annually.

(a) *[4 marks]*
What was the amount of each payment Henry received from the annuity?

(b) *[4 marks]*
At the end of 2007, just after the annuity's 14^{th} payment, Henry refunded it, that is, sold it back to the bank for its current value (value at the end of 2007). How much money did he receive?

(c) *[5 marks]*
At the beginning of 2008, Henry bought as many Acme bonds as he could with the money he received from the sale of the annuity. An Acme bond had a face value of \$1000, 6 semiannual coupons worth \$30 each until maturity at the end of 2010, and an **annual** yield rate of 7%. How many Acme bonds could Henry buy?

Name: ______________________________ Student #: ______________

B2. *[12 marks]*
Given: At the present time, 1800 people pay $9 each to take a commuter train to work. The number of people willing to ride the train at a price p is given by:

$$q = 600(6 - \sqrt{p})$$

(a) *[4 marks]*
At present, is the demand elastic or inelastic?

(b) *[2 marks]*
By examining $\dfrac{dr}{dp}$ at the present time, determine whether an increase in price will increase or decrease revenue (r).

(c) *[5 marks]*
Find the price that should be charged in order to maximize revenue.

(d) *[1 mark]*
At the price found in c), will an increase in price result in an increase or decrease in revenue? Why?

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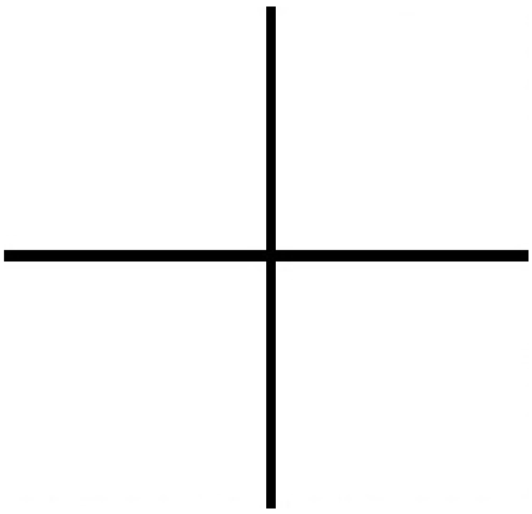
B3. *[10 marks]*

(a) *[4 marks]*

Sketch the functions

$$y = \sqrt{x-1} \text{ and } y = x - 1$$

on the axes below, labelling the points of intersection on the graph



(b) *[6 marks]*

Shade the region(s) between the graphs of the two functions from $x = 1$ to $x = 5$ and evaluate the area of the shaded figure.

[Express your final answer as a single number.]

Name: ________________________________ Student #: ____________________

B4. *[10 marks]*

Find $y(x)$ **explicitly in terms of x** such that $\dfrac{dy}{dx} = xe^{2y}$ and $y(0) = -1$.

[You may assume that x is always between $-e$ and e.]

Name: ______________________________ Student #: ______________________

B5. *[10 marks]*
Use the method of Lagrange multipliers to find all critical points of the problem

$$\begin{array}{rl} \text{maximize} & 4x+y \\ \text{subject to the contraint} & x^2+xy=27 \end{array}$$

[No marks will be given for any other method.] Make sure to specify the values of the Lagrange multiplier.

Name: ______________________________ Student #: ______________

Record your answers on the front page

PART A. MULTIPLE CHOICE

1. *[3 marks]*
If $PQ = R$

where $P = \begin{pmatrix} 1 & 5 & 7 & 8 & 9 \\ 3 & -1 & 0 & 4 & 2 \\ 2 & 4 & -2 & 1 & 6 \end{pmatrix}$

and $Q = \begin{pmatrix} 2 & 1 & 6 & 0 \\ 7 & -2 & 2 & -1 \\ 9 & 2 & 4 & 4 \\ 4 & 0 & 3 & 0 \\ 0 & 1 & 1 & 7 \end{pmatrix}$

then r_{24} is

A. 14

B. 15

C. 16

D. 17

E. undefined because P and Q cannot be multiplied together

$r_{24} = 3 \cdot 0 + (-1)(-1) + 0 \cdot 4 + 4 \cdot 0 + 2 \cdot 7 = 15$, B.

2. *[3 marks]*
If $f(x) = \begin{cases} x + e^k & \text{if } x \leq 1 \\ \dfrac{2x^2 - 2x}{x - 1} & \text{if } x > 1 \end{cases}$
is continuous at $x = 1$ then

A. $k = -1$

B. $k = 0$

C. $k = 1$

D. $k = 2$

E. there is no real number k such that f is continuous at $x = 1$

$\lim\limits_{x \to 1^-} f(x) = 1 + e^k = f(1)$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{2x^2 - 2x}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{2x(x-1)}{x - 1} = \lim\limits_{x \to 1^+} 2x = 2$. For continuity, we must therefore have $1 + e^k = 2$, so $e^k = 1$, so $k = 0$. B.

3. *[3 marks]*
If $y = x^{\ln x}$, then $y'(e) =$

A. $\dfrac{2}{e}$

B. $\dfrac{1}{e}$

C. e

D. 2

E. $\dfrac{1}{2e}$

$\ln y = \ln x \ln x = (\ln x)^2$

$\dfrac{1}{y}y' = 2\dfrac{\ln x}{x}$

$y' = \dfrac{2y\ln x}{x}$

When $x = e$, $y = e^{\ln e} = e^1 = e$

$y' = \dfrac{2e}{e}\ln e = 2$. D.

4. *[3 marks]*
The function $f(x) = xe^{-x}$ on the inverval $[-1, 2]$

A. has an absolute minimum at $x = -1$ and an absolute maximum at $x = 2$

B. has an absolute minimum at $x = -1$ and an absolute maximum at $x = 1$

C. has an absolute minimum at $x = 1$ and an absolute maximum at $x = 2$

D. has an absolute maximum at $x = 1$ and no absolute minimum

E. has an absolute minimum at $x = -1$ and no absolute maximum

$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$

f is continuous on $[-1, 2]$ and has only one critical point, at $x = 1$.

f must have an absolute maximum and an absolute minimum at $x = -1$, $x = 1$, or $x = 2$.

$f(-1) = -e$, $f(1) = \dfrac{1}{e}$, $f(2) = \dfrac{2}{e^2} < \dfrac{1}{e}$ (to see this, start with $2 < e$, and divide both sides by e^2). This means $f(1)$ is the maximum and $f(-1)$ is the minimum. B.

Name: ______________________ Student #: ______________

Record your answers on the front page

5. *[3 marks]*
If the demand equation is $p = \frac{q^3}{12} - 50q^2 + 10000q$, for what values of q is the **marginal revenue** $(\frac{dr}{dq})$ decreasing?
 A. $(0, 100)$
 B. $(100, 200)$
 C. $(200, \infty)$
 D. All values
 E. None

$r = pq = \frac{q^4}{12} - 50q^3 + 10,000q^2$

$\frac{dr}{dq} = \frac{q^3}{3} - 150q^2 + 20,000q$

For $\frac{dr}{dq}$ to decrease, we need <u>its</u> derivative to be negative.

$\frac{d^2r}{dq^2} = q^2 - 300q + 20,000 < 0$

$(q - 200)(q - 100) < 0$

	r''
$(-\infty, 100)$	+
$(100, 200)$	-
$(200, \infty)$	+

So $(100, 200)$. B.

6. *[3 marks]*
$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{x^3}$ is
 A. undefined
 B. 0
 C. 2
 D. $\frac{1}{6}$
 E. $\frac{4}{3}$

This limit has the indefinite form $\frac{0}{0}$ so L'Hopital's method applies.

$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{x^3} = \lim_{x \to 0} \frac{2e^{2x} - 2 - 4x}{3x^2}$ (note this is another $\frac{0}{0}$ form) $= \lim_{x \to 0} \frac{4e^{2x} - 4}{6x}$ (again of the form $\frac{0}{0}$) $= \lim_{x \to 0} \frac{8e^{2x}}{6} = \frac{8}{6} = \frac{4}{3}$. E.

7. *[3 marks]*
If Newton's method is used to estimate a solution to the equation

$$(x-2)^2 = \ln x$$

beginning with $x_1 = 1$, then x_2 is closest to

A. 0.67

B. 1.33

C. 1.39

D. 1.41

E. 1.43

Let $f(x) = (x-2)^2 - \ln x$

$f'(x) = 2(x-2) - \dfrac{1}{x}$

$f(1) = 1 - \ln 1 = 1$

$f'(1) = 2(-1) - 1 = -3$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1 - \dfrac{1}{-3} = 1 + \dfrac{1}{3} = \dfrac{4}{3}$. B.

8. *[3 marks]*

$$\int_1^e \frac{\ln x}{x} dx =$$

A. 2

B. 1

C. $\dfrac{1}{2}$

D. e

E. $\dfrac{1}{e}$

Let $u = \ln x$, $du = \dfrac{dx}{x}$

$\displaystyle\int_1^e \frac{\ln x}{x} dx = \int_0^1 u du = \frac{u^2}{2}\bigg|_0^1 = \frac{1}{2}$. C.

9. *[3 marks]*

$$\int \frac{x+2}{x^2-1}dx =$$

A. $(x+2)\ln|x^2-1|+C$

B. $\frac{5}{2}\ln|x^2-1|+C$

C. $\frac{1}{2}[\ln|x+2|-\ln|x+1|-\ln|x-1|]+C$

D. $\frac{1}{2}[\ln|x-1|-\ln|x+1|]+C$

E. $\frac{1}{2}[3\ln|x-1|-\ln|x+1|]+C$

Using partial fractions:

$\frac{x+2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$

$A(x+1)+B(x-1) = x+2.$

Method 1:

When $x=1$, we have $2A=3$ so $A=3/2$. When $x=-1$ we have $-2B=1$ so $B=-1/2$.

Method 2:

$(A+B)x+(A-B) = x+2$, so $A+B=1$ and $A-B=2$. Solve this system of equations for $A=3/2, B=-1/2$.

Either way, $\int \frac{x+2}{x^2-1} = \int \left[\frac{3}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}\right]dx$

$= \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|+C$. E.

10. *[3 marks]*

A town's population is expected to grow exponentially so that it will double every 50 years. If its population today is 10,000, then in 10 years its population will be

A. $12,039$

B. $11,755$

C. $10,821$

D. $11,487$

E. $10,432$

$P(t) = 10000e^{kt}$

$20000 = P(50) = 10000e^{50k}$

$e^{50k} = 2$

$e^k = 2^{\frac{1}{50}}$

$P(t) = 10000(2^{\frac{t}{50}})$

$P(10) = 10000(2^{\frac{1}{5}}) \approx 11487$. D.

11. *[3 marks]*

If $z = (x^2 + y^2)\ln(3x + y)$, then when $x = 0$ and $y = e$, $\dfrac{\partial z}{\partial x} =$

A. 0

B. 1

C. e^2

D. e

E. $3e$

$\dfrac{\partial z}{\partial x} = 2x\ln(3x + y) + \dfrac{x^2 + y^2}{3x + y} \cdot 3$

when $x = 0$ and $y = e$

$\dfrac{\partial z}{\partial x} = \dfrac{e^2}{e} \cdot 3 = 3e$. E.

12. *[3 marks]*

If $y^2 + z^2 = xyz + 1$, then when $x = y = z = 1$, $\dfrac{\partial z}{\partial y} =$

A. -1

B. 0

C. 1

D. 2

E. -2

Keeping x constant, $2y + 2z\dfrac{\partial z}{\partial y} = xz + xy\dfrac{\partial z}{\partial y}$.

At $x = y = z = 1$, $2 + 2\dfrac{\partial z}{\partial y} = 1 + \dfrac{\partial z}{\partial y}$

$\dfrac{\partial z}{\partial y} = -1$. A.

Record your answers on the front page

13. *[3 marks]*

If $x = 3r + 5s$, $y = 2r + 3s$, and $z = x^2 + y^2$, then when $r = s = 1$, $\dfrac{\partial z}{\partial r} =$

- **A**. 64
- **B**. 10
- **C**. 42
- **D**. 68
- **E**. 39

$\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r}$

$\dfrac{\partial x}{\partial r} = 3$, $\dfrac{\partial y}{\partial r} = 2$, $\dfrac{\partial z}{\partial x} = 2x = 16$ when $r = s = 1$, $\dfrac{\partial z}{\partial y} = 2y = 10$ when $r = s = 1$.

$\dfrac{\partial z}{\partial r} = 16 \cdot 3 + 10 \cdot 2 = 68$ when $r = s = 1$. D.

14. *[3 marks]*

The quantities q_A and q_B of goods A and B, which are sold when the respective unit prices of the goods are p_A and p_B, are

$q_A(p_A, p_B) = 200 - 5p_A + 10p_B - p_B^2$
$q_B(p_A, p_B) = 100 - 6p_A - 3p_B + p_A^2$

The two goods are complementary

- **A**. when $p_A > 3$ and $p_B < 5$
- **B**. when $p_A > 3$ and $p_B > 5$
- **C**. when $p_A < 3$ and $p_B > 5$
- **D**. for no values of p_A and p_B
- **E**. when $p_A < 3$ and $p_B < 5$

A and B are complementary when $\dfrac{\partial q_A}{\partial p_B} < 0$ and $\dfrac{\partial q_B}{\partial p_A} < 0$

$10 - 2p_B < 0$ and $-6 + 2p_A < 0$

$p_B > 5$ and $p_A < 3$. C.

15. *[3 marks]*
The critical point of

$$f(x, y) = x^2 + 2xy - y^2 - 2x - 6y + 3$$

is

A. $(2, -1)$, which is a saddle point

B. $(2, -1)$, which is a local max

C. $(2, -1)$, which is a local min

D. $(2, -1)$, but the second derivative test cannot determine its nature

E. not defined, because the partial derivatives are never both zero

$f_x = 2x + 2y - 2 = 0$

$f_y = 2x - 2y - 6 = 0$

First adding then subtracting these equations, we get

$\begin{array}{rr} 4x - 8 = 0 & x = 2 \\ 4y + 4 = 0 & y = -1 \end{array}$ So $(2, -1)$ is indeed critical.

$f_{xx} = 2, f_{yy} = -2, f_{xy} = f_{yx} = 2$

$D = f_{xx} f_{yy} - (f_{xy})^2 = -4 - 4 = -8 < 0$

$(2, -1)$ is a saddle point. A.

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PART B. WRITTEN-ANSWER QUESTIONS

B1. *[13 marks]*
At the end of December, 2000, Henry invested \$100,000 in an annuity which could be refunded at any time and which made 20 equal semiannual payments at the ends of June and December from 2001 to 2010 inclusive at 6% compounded semi-annually.

(a) *[4 marks]*
What was the amount of each payment Henry received from the annuity?

Let $R =$ semi-annual payment, $i = .03$ interest per half-year. Then $100,000 = Ra_{\overline{20|}.03}$.

So, $R = \dfrac{100,000 \cdot .03}{1-(1.03)^{-20}} = \$6721.57$.

(b) *[4 marks]*
At the end of 2007, just after the annuity's 14^{th} payment, Henry refunded it, that is, sold it back to the bank for its current value (value at the end of 2007). How much money did he receive?

He recieves the principal outstanding, namely, the value of the 6 remaining payments:

$Ra_{\overline{6|}.03} = 6721.57\dfrac{[1-(1.03)^{-6}]}{.03} = \$36,412$

(c) *[5 marks]*
At the beginning of 2008, Henry bought as many Acme bonds as he could with the money he received from the sale of the annuity. An Acme bond had a face value of \$1000, 6 semiannual coupons worth \$30 each until maturity at the end of 2010, and an **annual** yield rate of 7%. How many Acme bonds could Henry buy?

$P = V(1+i)^{-n} + rVa_{\overline{n|}i} = 1000(1.035)^{-6} + 30a_{\overline{6|}.035}$, the price of a single bond.

$P = \$973.36$ per \$1000 of face value.

$\dfrac{36,412}{973.36} = 37.4$. So Henry can buy 37 bonds.

Name: ________________ Student #: ________________

B2. *[12 marks]*

Given: At the present time, 1800 people pay \$9 each to take a commuter train to work. The number of people willing to ride the train at a price p is given by:

$$q = 600(6 - \sqrt{p})$$

(a) *[4 marks]*

At present, is the demand elastic or inelastic?

$$\eta = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q}\frac{dq}{dp} = \frac{p}{q} \cdot 600(-\frac{1}{2\sqrt{p}}) = -300\frac{\sqrt{p}}{q}$$

When $p = 9$ and $q = 1800$, $\eta = \dfrac{-300\sqrt{9}}{1800} = -\dfrac{1}{2}$

Demand is inelastic

(b) *[2 marks]*

By examining $\dfrac{dr}{dp}$ at the present time, determine whether an increase in price will increase or decrease revenue (r).

$r = pq = 600(6p - p^{3/2})$

$\dfrac{dr}{dp} = 600(6 - \dfrac{3}{2}p^{\frac{1}{2}})$. When $p = 9$

$\dfrac{dr}{dp} = 600(6 - \dfrac{9}{2}) = 900$, so an increase in price will increase revenue.

(c) *[5 marks]*

Find the price that should be charged in order to maximize revenue.

$\dfrac{dr}{dp} = 0$ when $6 = \dfrac{3}{2}p^{1/2} \Rightarrow p^{1/2} = 4 \Rightarrow p = \$16$.

$\dfrac{dr}{dp} = 300(4 - p^{1/2})$, so $\dfrac{d^2r}{dp^2} = -150p^{-\frac{1}{2}} < 0$ for all p.

So $\dfrac{dr}{dp} = 0$ gives the maximum. (Or: $\dfrac{dr}{dp} > 0$ when $p < 16$, and $\dfrac{dr}{dp} < 0$ when $p > 16$).

(d) *[1 mark]*

At the price found in c), will an increase in price result in an increase or decrease in revenue? Why?

An increase in price will decrease revenue, because, at $p = 16$, revenue is the most it can be. Or, because when $p > 16$, $\dfrac{dr}{dp} < 0$ so revenue decreases as p increases.

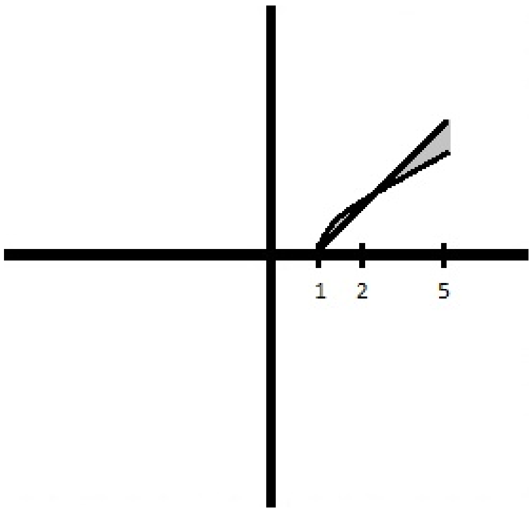
Name: ______________________________ Student #: ______________________

B3. *[10 marks]*

(a) *[4 marks]*
Sketch the functions

$$y = \sqrt{x-1} \text{ and } y = x - 1$$

on the axes below, labelling the points of intersection on the graph



Intersection is when $\sqrt{x-1} = x - 1$. Either $x = 1$ or, dividing by $\sqrt{x-1}$ (if $x \neq 1$), $1 = \sqrt{x-1}$ so $1 = x - 1$ so $2 = x$.

(b) *[6 marks]*
Shade the region(s) between the graphs of the two functions from $x = 1$ to $x = 5$ and evaluate the area of the shaded figure.

[Express your final answer as a single number.]

Area $= \displaystyle\int_1^2 \left[\sqrt{x-1} - (x-1)\right] dx + \int_2^5 \left[(x-1) - \sqrt{x-1}\right] dx$

$= \left[\frac{2}{3}(x-1)^{3/2} - \frac{(x-1)^2}{2}\right]_1^2 + \left[\frac{(x-1)^2}{2} - \frac{2}{3}(x-1)^{3/2}\right]_2^5 = \left[\frac{2}{3} - \frac{1}{2}\right] + \left[(\frac{16}{2} - \frac{16}{3}) - (\frac{1}{2} - \frac{2}{3})\right] =$
$\frac{1}{6} + \frac{16}{6} + \frac{1}{6} = \frac{18}{6} = 3$

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B4. *[10 marks]*

Find $y(x)$ **explicitly in terms of x** such that $\frac{dy}{dx} = xe^{2y}$ and $y(0) = -1$.

[You may assume that x is always between $-e$ and e.]

$\frac{dy}{dx} = xe^{2y}$: separate the variables

$e^{-2y}dy = xdx$

$\int e^{-2y}dy = \int xdx$

$\frac{e^{2y}}{-2} = \frac{x^2}{2} + C$

$e^{-2y} = -x^2 + K$

When $x = 0, y = -1$

$e^2 = K$

$e^{-2y} = e^2 - x^2$

$-2y = \ln(e^2 - x^2)$ (note that we are given $x^2 < e^2$ so $e^2 - x^2 > 0$).

$y = -\frac{1}{2}\ln(e^2 - x^2)$

or $y = \ln\left(\frac{1}{\sqrt{e^2 - x^2}}\right)$.

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B5. *[10 marks]*

Use the method of Lagrange multipliers to find all critical points of the problem

$$\begin{array}{ll} \text{maximize} & 4x+y \\ \text{subject to the contraint} & x^2+xy=27 \end{array}$$

[No marks will be given for any other method.] Make sure to specify the values of the Lagrange multiplier.

$L = 4x + y - \lambda(x^2 + xy - 27)$

$L_x = 4 - \lambda(2x + y), L_y = 1 - \lambda x, L_\lambda = -(x^2 + xy - 27).$

There are many ways to solve the equations $L_x = L_y = L_\lambda = 0$. Here's one:

$L_y = 0 \Rightarrow \lambda x = 1$, so $\lambda \neq 0$ and $x = \dfrac{1}{\lambda}$.

Then $L_x = 0 \Rightarrow 4 - 2 - \lambda y = 0$, so $\lambda y = 2$ and $y = \dfrac{2}{\lambda}$.

$L_\lambda = 0 \Rightarrow x^2 + xy = 27$, so $\dfrac{1}{\lambda^2} + \dfrac{2}{\lambda^2} = 27$, $\dfrac{3}{\lambda^2} = 27$, and we have $\lambda^2 = \dfrac{1}{9}$ so $\lambda = \pm\dfrac{1}{3}$.

$\lambda = \dfrac{1}{3} \Rightarrow x = 3$ and $y = 6$.

$\lambda = -\dfrac{1}{3} \Rightarrow x = -3$ and $y = -6$.

Slightly different is: $x = \dfrac{1}{\lambda}$ and $y = \dfrac{2}{\lambda} \Rightarrow y = 2x$

$x^2 + xy = 27 \Rightarrow 3x^2 = 27$

$x^2 = 9$

$x = \pm 3$

$x = 3 \Rightarrow y = 6$ and $\lambda = \dfrac{1}{3}$

$x = -3 \Rightarrow y = -6$ and $\lambda = -\dfrac{1}{3}$