FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL 2011

MAT 133Y1Y
Calculus and Linear Algebra for Commerce

Duration: 3 hours
Examiners: A. Igelfeld
P. Kergin
J. Tate
O. Yacobi

LEAVE	BLANK
Question	Mark
MC/45	
B1/12	
B2/10	
B3/11	
B4/11	
B5/11	
TOTAL	

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

NOTE:

1. **Aids Allowed:** A non-graphing calculator,with empty memory, to be supplied by student.
2. **Instructions:** Fill in the information on this page, and make sure your test booklet contains 14 pages.
3. This exam consists of 15 multiple choice questions, and 5 written-answer questions. For the **mutiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the front page** with your pencil. Each correct answer is worth 3 marks; a question left blank, or an incorrect answer or two answers for the same question is worth 0. For the **written-answer questions,** present your solutions in the space provided. The value of each written-answer question is indicated beside it.
4. Put your name and student number on each page of this examination.

ANSWER BOX FOR PART A
Circle the correct answer.

1.	**A.**	**B.**	C.	**D.**	**E.**
2.	**A.**	**B.**	C.	**D.**	**E.**
3.	**A.**	**B.**	C.	**D.**	**E.**
4.	**A.**	**B.**	C.	**D.**	**E.**
5.	**A.**	**B.**	C.	**D.**	**E.**
6.	**A.**	**B.**	C.	**D.**	**E.**
7.	**A.**	**B.**	C.	**D.**	**E.**
8.	**A.**	**B.**	C.	**D.**	**E.**
9.	**A.**	**B.**	C.	**D.**	**E.**
10.	**A.**	**B.**	C.	**D.**	**E.**
11.	**A.**	**B.**	C.	**D.**	**E.**
12.	**A.**	**B.**	C.	**D.**	**E.**
13.	**A.**	**B.**	C.	**D.**	**E.**
14.	**A.**	**B.**	C.	**D.**	**E.**
15.	**A.**	**B.**	C.	**D.**	**E.**

Name: ______________________________ Student #: ______________

Record your answers on the front page.

PART A. MULTIPLE CHOICE

1. *[3 marks]*

If $A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

and $AX = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$

then $X =$

A. $\begin{pmatrix} 2 & 3 \\ 6 & 8 \end{pmatrix}$

B. $\begin{pmatrix} -1 & 5 \\ -3 & 11 \end{pmatrix}$

C. $\begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$

D. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

E. $\begin{pmatrix} -1 & 7 \\ -2 & 10 \end{pmatrix}$

2. *[3 marks]*

If the demand function for a product is given by

$$p = 500e^{-q/20}$$

then the maximum value of revenue is

A. 500

B. 20

C. $1000/e$

D. $10,000$

E. $10,000/e$

Name: ______________________________ Student #: ______________

Record your answers on the front page.

3. *[3 marks]*

The function $f(x) = x^2 + \dfrac{2}{x}$ on the interval $[\frac{1}{3}, 2]$ has its maximum value

A. nowhere; there is no maximum

B. at $x = 1$

C. at $x = 2$

D. at $x = \frac{1}{3}$

E. at $x = 2^{-\frac{1}{3}}$

4. *[3 marks]*

The graph of $f(x) = e^x + e^{-x}$ is

A. increasing when $x > 0$ and always concave upward.

B. increasing when $x < 0$ and always concave upward.

C. increasing and concave upward everywhere.

D. increasing and concave downward everywhere.

E. increasing everywhere and concave upward when $x > 0$.

Name: ______________________________ Student #: ______________

Record your answers on the front page.

5. *[3 marks]*

If a country's savings (S) and national income (I) are related by: $2S^2 + I^2 = 3SI$ then when $I = 4$ and $S = 2$, the marginal propensity to save is:

A. $\frac{3}{4}$

B. $-\frac{8}{5}$

C. $\frac{5}{6}$

D. $\frac{1}{2}$

E. 2

6. *[3 marks]*

$$\lim_{x \to \infty} (x^2 + 2)^{\frac{1}{x^2+1}}$$

A. $= e$

B. $= 0$

C. $= 1$

D. $= -1$

E. does not exist

Name: ______________________________ Student #: ______________

Record your answers on the front page.

7. *[3 marks]*

If the demand for a certain product is determined by $q = 300 - 10p$ and the supply by $q = \dfrac{20p - 100}{3}$ where p is unit price and q is quantity then producers surplus is

A. 750

B. 500

C. 1250

D. 2000

E. 1000

8. *[3 marks]*

The average value of $f(x) = \dfrac{\ln x}{x}$ on the interval $[e, e^2]$ is

A. $\dfrac{\frac{1}{e} + \frac{1}{e^2}}{e^2 - e}$

B. $\dfrac{\frac{1}{e^2} - \frac{1}{e}}{e^2 - e}$

C. $\dfrac{3}{2(e^2 - e)}$

D. $\dfrac{e^2 + e}{e^2 - e}$

E. $\dfrac{1}{2(e^2 - e)}$

Name: ____________________________ Student #: _______________

Record your answers on the front page.

9. *[3 marks]*

The present value of a continuous annuity at an annual rate of 9% compounded continuously for 5 years, if the payment at time t is at the annual rate of $30,000, is closest to

A. $\$98,000$

B. $\$117,000$

C. $\$118,000$

D. $\$120,000$

E. $\$121,000$

10. *[3 marks]*

Let $k > 0$ be a constant. Then
$\int_1^\infty ke^{-kx}dx$ is

A. e^{-k}

B. $-e^{-k} - 1$

C. 1

D. $-e^{-k}$

E. divergent, i.e. the integral diverges.

Name: ______________________________ Student #: ______________

Record your answers on the front page.

11. *[3 marks]*

If two goods have unit prices $p_1 > 0$ and $p_2 > 0$, and their respective demands are

$$q_1(p_1, p_2) = 400 - 6p_1 + p_1^2 + 4p_2 - p_2^2$$

$$q_2(p_1, p_2) = 500 - p_1 - 4p_1^2 - 2p_2 - 3p_2^2$$

For which p_1 and p_2 are the goods complementary?

A. $p_2 > 2$, any p_1

B. for no values of p_1 and p_2

C. $p_1 < 3$, any p_2

D. $p_2 < 2$, any p_1

E. $p_1 > 3$, any p_2

12. *[3 marks]*

If $f(x, y, z) = e^{2xy+3z}$, then $f_{xyz}(1, 1, 1) =$

A. $15e^5$

B. $16e^5$

C. $12e^5$

D. $18e^5$

E. $20e^5$

Name: ______________________________ Student #: ______________

Record your answers on the front page.

13. *[3 marks]*

If $3x^2yz + 1 = 2x^2 + y^2 + z^2$ defines y implicitly as a function of x and z, then when $(x, y, z) = (1, 1, 2)$, $\dfrac{\partial y}{\partial x} =$

A. 1

B. -2

C. 0

D. -1

E. 2

14. *[3 marks]*

If $x = r^2 + s^2$, $y = rs$ and $z = f(x, y)$ has constant partial derivatives $\dfrac{\partial z}{\partial x} = 3$ and $\dfrac{\partial z}{\partial y} = -1$, then when $r = 2$ and $s = 5$, $\dfrac{\partial z}{\partial r} =$

A. 5

B. 8

C. 4

D. 6

E. 7

Record your answers on the front page.

15. *[3 marks]*

The function $f(x, y) = xy + 3e^{-x}$ has

A. a local minimum and a local maximum

B. a local minimum but no local maximum

C. a local maximum but no local minimum

D. no local maximum and no local minimum

E. 2 local maxima and 1 local minimum

Name: ________________________ Student #: ______________

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[12 marks]*

(a) *[6 marks]*

A \$300,000 mortgage is to be repaid by making equal monthly payments for 15 years, the first payment 1 month after the loan is granted. If interest is 8% per year compounded semiannually find (to within \$0.01) the amount of each payment.

(b) *[6 marks]*

A \$40,000 debt with interest at 6% per year compounded monthly is to be repaid by making payments at the end of each month for 8 years. The payments are all of the same size (X dollars each) with 2 exceptions: the 36^{th} payment is to be $10X$ dollars and the last payment is to be \$5,000. To within \$0.01, find X.

Name: ______________________________ Student #: ______________

B2. *[10 marks]* Find the area between the curves $y = xe^x$ and $y = -x$ from $x = -2$ to $x = 1$.

Name: ______________________________ Student #: ______________

B3. *[11 marks]*

(a) *[7 marks]*

[Here, give your final answer to 3 decimal places.]

Find $\displaystyle\int_3^4 \frac{dx}{x(x-1)(x-2)}$

(b) *[4 marks]*

[Here, give your final answer to 3 decimal places, or show that the integral diverges.]

What happens if the limits of integration of the integral in (a) are changed to $\displaystyle\int_3^\infty \frac{dx}{x(x-1)(x-2)}$?

Name: ______________________ Student #: ______________

B4. *[11 marks]*

Solve the following problems showing all your work:

(a) *[5 marks]*

If $\dfrac{dy}{dx} = 3x^2e^y + 2xe^y + e^y$ and $y(0) = 0$, find y explicitly as a function of x.

(b) *[6 marks]*

If $\dfrac{dp}{dq} = \dfrac{e^q\sqrt{1+p^2}}{p}$ and $p = \sqrt{3}$ when $q = 0$, what is p when $q = 1$? You may assume p is positive.

Name: ____________________________ Student #: ______________

B5. *[11 marks]*

The production function for a certain factory is given by $P(l, k) = 200l^{1/4}k^{3/4}$ where l is the number of units of labour and k is the number of units of capital. Labour costs \$20/unit and capital costs \$30/unit and the total amout spent on labour and capital is \$16,000.

By using the method of Lagrange multipliers find the number of units of labour and capital that maximize production.

[No marks will be given for any method except Lagrange multipliers.]

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ANSWER BOX FOR PART A
Circle the correct answer.

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PART A. MULTIPLE CHOICE

1. *[3 marks]*

If $A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

and $AX = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$

then $X =$

A. $\begin{pmatrix} 2 & 3 \\ 6 & 8 \end{pmatrix}$

B. $\begin{pmatrix} -1 & 5 \\ -3 & 11 \end{pmatrix}$

C. $\begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$

D. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

E. $\begin{pmatrix} -1 & 7 \\ -2 & 10 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1}AX = IX &= X \\ A^{-1}\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} &= X \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} &= X \\ \begin{pmatrix} -1 & 5 \\ -3 & 11 \end{pmatrix} &= X \end{aligned}$$

2. *[3 marks]*

If the demand function for a product is given by

$$p = 500e^{-q/20}$$

then the maximum value of revenue is

A. 500

B. 20

C. $1000/e$

D. $10,000$

E. $10,000/e$

Solution

$$\begin{aligned} R &= pq = 500qe^{-q/20} \\ \frac{dR}{dq} &= 500(e^{-q/20} - \frac{q}{20}e^{-q/20}) \\ &= 500e^{-q/20}(1 - \frac{q}{20}) \\ \frac{dR}{dq} &= 0 \text{ when } q = 20 \\ \frac{dR}{dq} &< 0 \text{ when } q > 20 \\ \frac{dR}{dq} &> 0 \text{ when } q < 20 \\ R & \quad \text{max at } q = 20 \\ R &= 500 \cdot 20e^{-1} = \frac{10,000}{e} \end{aligned}$$

Name: ________________________________ Student #: ________________

Record your answers on the front page.

3. *[3 marks]*

The function $f(x) = x^2 + \dfrac{2}{x}$ on the interval $[\frac{1}{3}, 2]$ has its maximum value

A. nowhere; there is no maximum

B. at $x = 1$

C. at $x = 2$

D. at $x = \frac{1}{3}$

E. at $x = 2^{-\frac{1}{3}}$

Solution f is continuous on $[\frac{1}{3}, 2]$ so must have a maximum.

$$f'(x) = 2x - \frac{2}{x^2} = \frac{2}{x^2}(x^3 - 1)$$

The only critical point is at $x = 1$.

$$\begin{aligned} f(\frac{1}{3}) &= \frac{1}{9} + 6 \qquad \text{max} \\ f(2) &= 5 \\ f(1) &= 3 \end{aligned}$$

4. *[3 marks]*

The graph of $f(x) = e^x + e^{-x}$ is

A. increasing when $x > 0$ and always concave upward.

B. increasing when $x < 0$ and always concave upward.

C. increasing and concave upward everywhere.

D. increasing and concave downward everywhere.

E. increasing everywhere and concave upward when $x > 0$.

Solution

$$f'(x) = e^x - e^{-x} = e^{-x}(e^{2x} - 1) \quad \begin{array}{l} > 0 \text{ when } x > 0 \text{ only} \\ < 0 \text{ when } x < 0 \text{ only} \end{array}$$

is already the only possible answer.

$$f''(x) = e^x + e^{-x} > 0 \text{ for all } x$$

so concave upward.

Name: ______________________ Student #: ______________

Record your answers on the front page.

5. *[3 marks]*

If a country's savings (S) and national income (I) are related by: $2S^2 + I^2 = 3SI$ then when $I = 4$ and $S = 2$, the marginal propensity to save is:

A. $\frac{3}{4}$

B. $-\frac{8}{5}$

C. $\frac{5}{6}$

D. $\frac{1}{2}$

E. 2

Solution

Marginal propensity to save is $\frac{dS}{dI}$.

$$\begin{aligned} 4S\frac{dS}{dI} + 2I &= 3\frac{dS}{dI}I + 3S \\ \text{at } I = 4, S = 2 \qquad 8\frac{dS}{dI} + 8 &= 12\frac{dS}{dI} + 6 \\ 2 &= 4\frac{dS}{dI} \\ \frac{dS}{dI} &= \frac{1}{2} \end{aligned}$$

6. *[3 marks]*

$$\lim_{x\to\infty} (x^2+2)^{\frac{1}{x^2+1}}$$

A. $= e$

B. $= 0$

C. $= 1$

D. $= -1$

E. does not exist

Solution

$$\begin{aligned} \ln(y) &= \frac{\ln(x^2+2)}{x^2+1} \qquad \frac{\infty}{\infty} \\ \lim_{x\to\infty} \ln(y) &= \lim_{x\to\infty} \frac{\frac{2x}{x^2+2}}{2x} \\ &= \lim_{x\to\infty} \frac{1}{x^2+2} = 0 \\ \ln(y) &\to 0 \\ y = e^{\ln(y)} &\to e^0 = 1 \end{aligned}$$

Name: ______________________________ Student #: ______________

Record your answers on the front page.

7. *[3 marks]*

If the demand for a certain product is determined by $q = 300 - 10p$ and the supply by $q = \dfrac{20p - 100}{3}$ where p is unit price and q is quantity then producers surplus is

A. 750
B. 500
C. 1250
D. 2000
E. 1000

Solution

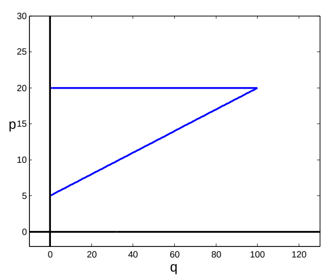

Figure 1:

Equilibrium:

$$\begin{aligned} 300 - 10p &= \frac{20p - 100}{3} \\ 900 - 30p &= 20p - 100 \\ 1000 &= 50p \\ 20 &= p \\ q &= 100 \end{aligned}$$

When $q = 0$, $p = 5$ on the supply curve.

$$\begin{aligned} PS &= \frac{1}{2} \cdot (20 - 5) \cdot 100 \\ &= 750 \end{aligned}$$

by triangle area.

Alternate Solution 1

$$\begin{aligned} PS &= \int_0^{100} \left(\frac{3q + 100}{20} - 5 \right) dq \\ &= \int_0^{100} \frac{3q}{20} dq \\ &= \left. \frac{3q^2}{40} \right|_0^{100} \\ &= \frac{30,000}{40} \\ &= 750 \end{aligned}$$

Alternate Solution 2

$$\begin{aligned} PS &= \int_5^{20} \frac{20p - 100}{3} dp \\ &= \left. \left(\frac{10p^2}{3} - \frac{100p}{3} \right) \right|_5^{20} \\ &= 750 \end{aligned}$$

8. *[3 marks]*

The average value of $f(x) = \dfrac{\ln x}{x}$ on the interval $[e, e^2]$ is

A. $\dfrac{\frac{1}{e} + \frac{1}{e^2}}{e^2 - e}$

B. $\dfrac{\frac{1}{e^2} - \frac{1}{e}}{e^2 - e}$

C. $\dfrac{3}{2(e^2 - e)}$

D. $\dfrac{e^2 + e}{e^2 - e}$

E. $\dfrac{1}{2(e^2 - e)}$

Solution

$$\begin{aligned} Avf &= \frac{1}{e^2 - e} \int_e^{e^2} \frac{\ln x}{x} dx \qquad u = \ln x, du = \frac{dx}{x} \\ &= \frac{1}{e^2 - e} \int_1^2 u du \\ &= \frac{1}{e^2 - e} \left. \frac{u^2}{2} \right|_1^2 \\ &= \frac{1}{e^2 - e} \frac{4 - 1}{2} \\ &= \frac{3}{2(e^2 - e)} \end{aligned}$$

Name: ________________________ Student #: ______________

Record your answers on the front page.

9. *[3 marks]*

The present value of a continuous annuity at an annual rate of 9% compounded continuously for 5 years, if the payment at time t is at the annual rate of $\$30,000$, is closest to

A. $\$98,000$

B. $\$117,000$

C. $\$118,000$

D. $\$120,000$

E. $\$121,000$

Solution

$$\begin{aligned} PV &= \int_0^5 30,000e^{-0.09t}dt \\ &= \frac{30,000}{-0.09}e^{-0.09t}\Big|_0^5 \\ &= \frac{30,000}{0.09}(1-e^{-0.45}) \\ &= 120,790.62 \end{aligned}$$

10. *[3 marks]*

Let $k > 0$ be a constant. Then $\int_1^\infty ke^{-kx}dx$ is

A. e^{-k}

B. $-e^{-k}-1$

C. 1

D. $-e^{-k}$

E. divergent, i.e. the integral diverges.

Solution

$$\begin{aligned} \int_1^\infty ke^{-kx}dx &= \lim_{R\to\infty}\int_1^R ke^{-kx}dx \\ &= \lim_{R\to\infty} -e^{-kx}\Big|_1^R \\ &= \lim_{R\to\infty}(e^{-k}-e^{-kR}) \\ &= e^{-k} \end{aligned}$$

Name: ______________________ Student #: ______________

Record your answers on the front page.

11. *[3 marks]*

If two goods have unit prices $p_1 > 0$ and $p_2 > 0$, and their respective demands are

$$\begin{aligned} q_1(p_1, p_2) &= 400 - 6p_1 + p_1^2 + 4p_2 - p_2^2 \\ q_2(p_1, p_2) &= 500 - p_1 - 4p_1^2 - 2p_2 - 3p_2^2 \end{aligned}$$

For which p_1 and p_2 are the goods complementary?

A. $p_2 > 2$, any p_1

B. for no values of p_1 and p_2

C. $p_1 < 3$, any p_2

D. $p_2 < 2$, any p_1

E. $p_1 > 3$, any p_2

Solution

$$\begin{aligned} \text{i.e. } \frac{\partial q_1}{\partial p_2} &< 0 \text{ and } \frac{\partial q_2}{\partial p_1} < 0. \\ \frac{\partial q_1}{\partial p_2} &= 4 - 2p_2 < 0 \text{ only if } p_2 > 2 \\ \frac{\partial q_2}{\partial p_1} &= -1 - 8p_1 < 0 \text{ always} \end{aligned}$$

12. *[3 marks]*

If $f(x, y, z) = e^{2xy+3z}$, then $f_{xyz}(1, 1, 1) =$

A. $15e^5$

B. $16e^5$

C. $12e^5$

D. $18e^5$

E. $20e^5$

Solution

Mixed partials are equal, so ok to do $\frac{\partial}{\partial z}$ first.

$$\begin{aligned} f_z &= 3e^{2xy+3z} \\ f_{zx} &= 3e^{2xy+3z}2y = 6ye^{2xy+3z} \\ f_{xyz} &= f_{zxy} = 6e^{2xy+3z} + 6ye^{2xy+3z}2x \end{aligned}$$

at $(1, 1, 1)$ we get $f_{xyz} = 6e^5 + 12e^5 = 18e^5$.

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13. *[3 marks]*

If $3x^2yz + 1 = 2x^2 + y^2 + z^2$ defines y implicitly as a function of x and z, then when $(x, y, z) = (1, 1, 2)$, $\quad \dfrac{\partial y}{\partial x} =$

A. 1

B. -2

C. 0

D. -1

E. 2

Solution

$$\begin{aligned} 6xyz + 3x^2\frac{\partial y}{\partial x}z &= 4x + 2y\frac{\partial y}{\partial x} \\ \text{At } (1,1,2), \quad 12 + 3\frac{\partial y}{\partial x}2 &= 4 + 2\frac{\partial y}{\partial x} \\ 4\frac{\partial y}{\partial x} &= -8 \\ \frac{\partial y}{\partial x} &= -2 \end{aligned}$$

14. *[3 marks]*

If $x = r^2 + s^2$, $y = rs$ and $z = f(x, y)$ has constant partial derivatives $\dfrac{\partial z}{\partial x} = 3$ and $\dfrac{\partial z}{\partial y} = -1$, then when $r = 2$ and $s = 5$, $\quad \dfrac{\partial z}{\partial r} =$

A. 5

B. 8

C. 4

D. 6

E. 7

Solution

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= 3 \cdot 2r + (-1)s \\ \text{At } r = 2, s = 5 \quad \frac{\partial z}{\partial r} &= 12 - 5 = 7 \end{aligned}$$

Name: ______________________ Student #: ______________

Record your answers on the front page.

15. *[3 marks]*

The function $f(x, y) = xy + 3e^{-x}$ has

A. a local minimum and a local maximum

B. a local minimum but no local maximum

C. a local maximum but no local minimum

D. no local maximum and no local minimum

E. 2 local maxima and 1 local minimum

Solution

$$\begin{aligned} f_x &= y - 3e^{-x} = 0 \\ f_y &= x = 0 \end{aligned}$$

Critical point: $x = 0$, so $y = 3e^{-0} = 3$.

$$\begin{aligned} f_{xx} &= 3e^{-x} \\ f_{yy} &= 0 \\ f_{xy} &= f_{yx} = 1 \\ D &= f_{xx}f_{yy} - f_{xy}^2 = -1 \text{ always} \end{aligned}$$

There are no local extrema.

Name: ______________________________ Student #: ______________

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[12 marks]*

(a) *[6 marks]*

A $300,000 mortgage is to be repaid by making equal monthly payments for 15 years, the first payment 1 month after the loan is granted. If interest is 8% per year compounded semiannually find (to within $0.01) the amount of each payment.

Solution

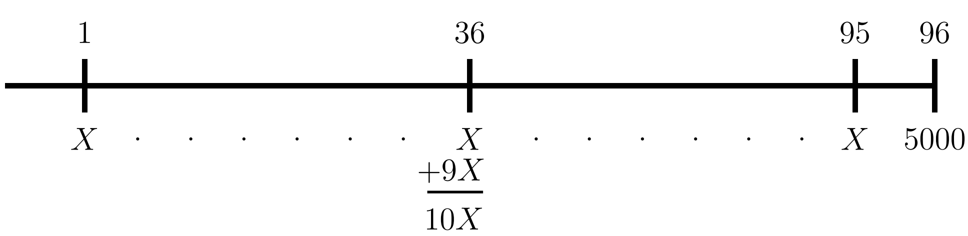
$$\begin{aligned}(1+i)^{12} &= 1.04^2 \\ 300,000 &= Ra_{\overline{180}|i} \\ R &= \frac{300,000}{a_{\overline{180}|i}} = \frac{300,000i}{1-(1+i)^{-180}} \\ &= 300,000\frac{1.04^{\frac{1}{6}}-1}{1-1.04^{-30}} = \$2844.46\end{aligned}$$

(b) *[6 marks]*

A $40,000 debt with interest at 6% per year compounded monthly is to be repaid by making payments at the end of each month for 8 years. The payments are all of the same size (X dollars each) with 2 exceptions: the 36^{th} payment is to be $10X$ dollars and the last payment is to be $5,000. To within $0.01, find X.

Solution

$$i = 0.005 \text{ per month}$$



$$\begin{aligned}40,000 &= Xa_{\overline{95}|0.005} + 9X(1.005)^{-36} + 5000(1.005)^{-96} \\ X &= \frac{40,000 - 5000(1.005)^{-96}}{a_{\overline{95}|0.005} + 9(1.005)^{-36}} \\ X &= \$444.63\end{aligned}$$

Name: ______________________________ Student #: ______________

B2. *[10 marks]* Find the area between the curves $y = xe^x$ and $y = -x$ from $x = -2$ to $x = 1$.

Solution

$y = xe^x$ and $y = -x$ intersect only when

$$\begin{aligned} xe^x &= -x \\ xe^x + x &= 0 \\ x(e^x + 1) &= 0 \\ x &= 0 \text{ only} \end{aligned}$$

We need to know which functions is above and which is below. Since both functions are continuous, they can only change places, if at all, at $x = 0$. On $[-2, 0]$ if we test at $x = -1$

$$y = xe^x = -e^{-1} = -\frac{1}{e} \text{ and } y = -x = 1$$

so $y = -x$ lies above $y = xe^x$. On $[0, 1]$ $xe^x > 0$ but $-x < 0$ so $y = xe^x$ lies above $y = -x$ (we could have used this reasoning on $[-2, 0]$ also). Hence:

$$\begin{aligned} \text{Area} &= \int_{-2}^{0} (-x - xe^x)dx + \int_{0}^{1} (xe^x - (-x))dx \\ \text{Now,} & \quad \int xe^x dx \qquad u = x, dv = e^x dx, du = dx, v = e^x \\ &= xe^x - \int e^x dx = xe^x - e^x \\ \text{So Area} &= \left[-\frac{x^2}{2} - xe^x + e^x\right]_{-2}^{0} + \left[xe^x - e^x + \frac{x^2}{2}\right]_{0}^{1} \\ &= [1 - (\frac{-4}{2} + 2e^{-2} + e^{-2})] + [(e - e + \frac{1}{2}) - (-1)] \\ &= 3 - 3e^{-2} + \frac{3}{2} = \frac{9}{2} - \frac{3}{e^2} \\ &\approx 4.094 \end{aligned}$$

Name: ________________________________ Student #: ______________

B3. *[11 marks]*

(a) *[7 marks]*

[Here, give your final answer to 3 decimal places.]

Find $\displaystyle\int_3^4 \frac{dx}{x(x-1)(x-2)}$

Solution

$$\begin{aligned}
\frac{1}{x(x-1)(x-2)} &= \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-2} \\
A(x-1)(x-2)+Bx(x-2)+Cx(x-1) &= 1 \\
x=0 &\Rightarrow 2A=1 \quad A=\frac{1}{2} \\
x=1 &\Rightarrow -B=1 \quad B=-1 \\
x=2 &\Rightarrow 2C=1 \quad C=\frac{1}{2} \\
&\int_3^4 \left(\frac{1}{2x}-\frac{1}{x-1}+\frac{1}{2(x-2)}\right)dx \\
&= \left[\frac{1}{2}\ln x-\ln|x-1|+\frac{1}{2}\ln|x-2|\right]_3^4 \\
&= \frac{1}{2}\left[\ln\left|\frac{x(x-2)}{(x-1)^2}\right|\right]_3^4 \\
&= \frac{1}{2}\left[\ln\frac{8}{9}-\ln\frac{3}{4}\right]=\frac{1}{2}\ln\frac{32}{27} \\
&\approx 0.085
\end{aligned}$$

(b) *[4 marks]*

[Here, give your final answer to 3 decimal places, or show that the integral diverges.]

What happens if the limits of integration of the integral in (a) are changed to $\displaystyle\int_3^\infty \frac{dx}{x(x-1)(x-2)}$?

Solution

$$\begin{aligned}
\lim_{R\to\infty}\int_3^R &= \lim_{R\to\infty}\frac{1}{2}\ln\left(\frac{R(R-2)}{(R-1)^2}\right)-\frac{1}{2}\ln\frac{3}{4} \\
\text{But}\quad \frac{R(R-2)}{(R-1)^2} &\to 1 \text{ as } R\to\infty \\
\text{so}\quad \ln\frac{R(R-2)}{(R-1)^2} &\to 0 \\
\int &= -\frac{1}{2}\ln\frac{3}{4}\approx 0.144
\end{aligned}$$

Name: ________________________________ Student #: ________________

B4. *[11 marks]*

Solve the following problems showing all your work:

(a) *[5 marks]*

If $\frac{dy}{dx} = 3x^2e^y + 2xe^y + e^y$ and $y(0) = 0$, find y explicitly as a function of x.

Solution

$$\begin{aligned}
e^{-y}dy &= (3x^2 + 2x + 1)dx \\
\text{Integrating} \quad -e^{-y} &= x^3 + x^2 + x + C \\
\text{At } x = 0, y = 0 : -e^0 &= C \Rightarrow C = -1 \\
-e^{-y} &= x^3 + x^2 + x - 1 \\
e^{-y} &= 1 - x - x^2 - x^3 \\
-y &= \ln(1 - x - x^2 - x^3) \\
y &= -\ln(1 - x - x^2 - x^3)
\end{aligned}$$

(b) *[6 marks]*

If $\frac{dp}{dq} = \frac{e^q\sqrt{1+p^2}}{p}$ and $p = \sqrt{3}$ when $q = 0$, what is p when $q = 1$? You may assume p is positive.

Solution

$$\begin{aligned}
\int \frac{pdp}{\sqrt{1+p^2}} &= \int e^q dq = e^q + C \\
(1+p^2)^{\frac{1}{2}} &= e^q + C \qquad \text{At } p = \sqrt{3}, q = 0 \\
4^{\frac{1}{2}} &= e^0 + C \qquad \text{so C} = 1 \\
(1+p^2)^{\frac{1}{2}} &= e^q + 1 \\
\text{when} \quad q = 1 & \\
(1+p^2)^{\frac{1}{2}} &= e + 1 \\
1 + p^2 &= (e+1)^2 \\
p^2 &= e^2 + 2e \\
p &= \sqrt{e^2 + 2e} \quad \text{because } p > 0 \\
p &\approx 3.58
\end{aligned}$$

Name: ________________________ Student #: ________________

B5. *[11 marks]*

The production function for a certain factory is given by $P(l, k) = 200l^{1/4}k^{3/4}$ where l is the number of units of labour and k is the number of units of capital. Labour costs \$20/unit and capital costs \$30/unit and the total amout spent on labour and capital is \$16,000.

By using the method of Lagrange multipliers find the number of units of labour and capital that maximize production.

[No marks will be given for any method except Lagrange multipliers.]

Solution

$$\begin{aligned}
L &= 200l^{\frac{1}{4}}k^{\frac{3}{4}} - \lambda(20l + 30k - 16,000) \\
\frac{\partial L}{\partial l} &= 50l^{-\frac{3}{4}}k^{\frac{3}{4}} - 20\lambda = 0 \\
\frac{\partial L}{\partial k} &= 150l^{\frac{1}{4}}k^{-\frac{1}{4}} - 30\lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= 20l + 30k - 16,000 = 0 \\
5\left(\frac{k}{l}\right)^{\frac{3}{4}} &= 2\lambda \qquad \text{from the 1st equation} \\
10\left(\frac{l}{k}\right)^{\frac{1}{4}} &= 2\lambda \qquad \text{from the 2nd equation}
\end{aligned}$$

Dividing the 2nd equation by the first

$$2\frac{l}{k} = 1 \text{ so } k = 2l$$

subbing into the $\frac{\partial L}{\partial \lambda}$ equation (or the constraint)

$$\begin{aligned}
20l + 60l &= 16,000 \\
80l &= 16,000 \\
l &= 200 \\
k &= 400
\end{aligned}$$