FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2010

MAT 133Y1Y
Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. *[3 marks]*
In any solution of the system

$$\begin{aligned} w - 2x + 2y - z &= 8 \\ 2w - 4x + 3y - z &= 13 \end{aligned}$$

we must have $w =$

A. $x + 2z - 2$

B. $-y + 2z - 3$

C. $2y - z + 4$

D. $2x - z + 2$

E. $-x + z + 3$

2. *[3 marks]*
If $A = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, then $AB^{-1}A^{\top} =$

A. $\begin{bmatrix} 2 \end{bmatrix}$

B. $\begin{bmatrix} -1 \end{bmatrix}$

C. $\begin{bmatrix} -2 \end{bmatrix}$

D. $\begin{bmatrix} 0 \end{bmatrix}$

E. $\begin{bmatrix} 1 \end{bmatrix}$

3. *[3 marks]*
The equation of the tangent line to the curve $x^3 + xy^2 - y^3 + 3 = 0$ at the point $(1, 2)$ is

A. $7y - 8x - 9 = 0$

B. $7y - 8x - 6 = 0$

C. $7x + 8y + 23 = 0$

D. $2x + y + 4 = 0$

E. $7x - 8y + 9 = 0$

4. *[3 marks]*

If $f(x) = e^x \sqrt{\frac{x+4}{x+1}}$, then $f'(0) =$

A. $\frac{5}{4}$

B. $-\frac{3}{4}$

C. $\frac{5}{8}$

D. $\frac{13}{4}$

E. $-\frac{3}{8}$

5. *[3 marks]*

$\lim_{x \to 0} \frac{1}{x} - \frac{1}{e^x - 1} =$

A. $-\frac{1}{2}$

B. $-\frac{1}{4}$

C. $\frac{1}{2}$

D. 1

E. -1

6. *[3 marks]*

If Newton's method is used to find a root of the equation $4x^3 - 36x^2 + 77x - 40 = 0$ with initial trial $x_1 = 1$, then, rounded to two decimal places, $x_2 =$

A. 1.00

B. 0.71

C. 0.41

D. 0.79

E. 0.77

7. *[3 marks]*

The absolute maximum value and absolute minimum value of $f(x) = 2x^3 - 3x^2 - 12x + 15$ on the interval $[0, 3]$ are

A. absolute maximum 22, absolute minimum -5

B. absolute maximum 6, absolute minimum -5

C. absolute maximum 15, absolute minimum -5

D. absolute maximum 21, absolute minimum 0

E. no absolute maximum, absolute minimum -5

8. *[3 marks]*
A manufacturer's marginal revenue function is

$$\frac{dr}{dq} = \frac{200}{\sqrt{q+4}}$$

where the revenue r is in dollars and q represents units of production.

The change in the manufacturer's total revenue (in dollars) if production is increased from 60 units to 140 units is

A. -2400

B. $+1200$

C. $+1600$

D. -1600

E. $+2400$

9. *[3 marks]*
The rate of interest is 3% per year compounded continuously. The continuous rate of cash flow at time t is $100e^{-0.02t}$ dollars per year. After 10 years, the accumulated amount (to the nearest dollar) is

A. \$ 276

B. \$1030

C. \$ 53

D. \$ 234

E. \$1062

10. *[3 marks]*
The average value of $f(x) = \dfrac{1}{x}$ on the interval $[1, e]$ is

A. $\dfrac{1}{e}$

B. $\dfrac{1}{e-1}$

C. 1

D. $\dfrac{e+1}{2(e-1)}$

E. e

11. *[3 marks]*
$\displaystyle\int_0^1 xe^x\,dx =$

A. $\dfrac{e}{2}$

B. e

C. $\dfrac{1}{2}$

D. 1

E. -1

12. *[3 marks]*

Given: $f(x,y) = y^2e^{2x} + x^3 \ln\left(\dfrac{y}{x}\right)$,

$f_x(1,e) =$

A. $e^4 + 1$

B. $e^4 + \dfrac{1}{e} + 1$

C. $4e^3 - 3e + 3$

D. $2e^4 - 3$

E. $2e^4 + 2$

13. *[3 marks]*

The joint demand functions for the products A and B are given by:

$$q_A = \frac{200}{p_A\sqrt{p_B}} \qquad q_B = \frac{300}{p_B\sqrt[3]{p_A}}$$

Which of the following statements are true?

A. $\dfrac{\partial q_A}{\partial p_A} > 0$

B. $\dfrac{\partial q_B}{\partial p_B} > 0$

C. A and B are competitive products

D. A and B are complementary products

E. A and B are neither competitive nor complementary products

14. *[3 marks]*

Let $z = f(x,y)$ while $x = g(r,s)$ and $y = h(r,s)$.

When $r = 2$ and $s = 3$, $x = 1$ and $y = 2$.

Also,

$$\begin{array}{ll} f_x(1,2) = 4 & f_x(2,3) = -1 \\ f_y(1,2) = 5 & f_y(2,3) = 1 \\ g_r(1,2) = 2 & g_r(2,3) = 3 \\ h_r(1,2) = 0 & h_r(2,3) = -2 \end{array}$$

Then, when $r = 2$ and $s = 3$, we have $\dfrac{\partial z}{\partial r} =$

A. 2

B. -5

C. -2

D. 12

E. 8

15. *[3 marks]*
$f(x, y) = 2x^2 - 8x + 3y^2 - 2y^3$ has

A. a relative minimum at $(2, 0)$ and a relative maximum at $(2, 1)$

B. a relative maximum at $(2, 0)$ and a relative minimum at $(2, 1)$

C. a relative maximum at $(2, 0)$ and no relative minimum

D. a relative minimum at $(2, 0)$ and no relative maximum

E. no relative extrema

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[12 marks]*

(a) *[4 marks]*
A certain bond has a face value of \$100, 9 years to maturity, *semiannual* interest payments of \$2 each, and *annual* yield 5%. What is its market price?

Parts (b) and (c) concern a $\$400,000$ mortgage payable over 15 years by equal monthly payments, with 7% interest compounded semiannually.

(b) *[4 marks]*
What is the amount of each payment?

(c) *[4 marks]*
How much interest is included in the last payment of the 10^{th} year of the mortgage?

B2. *[13 marks]*
Given: $f(x) = xe^{x/2}$.

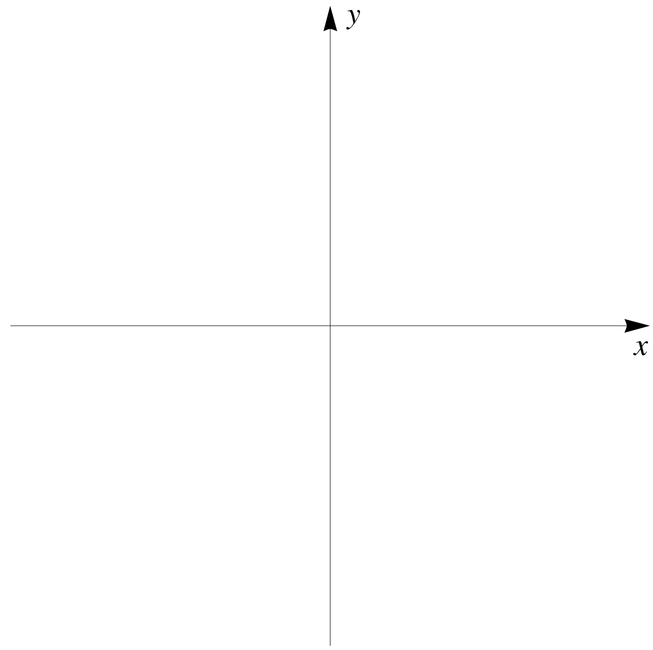
(a) *[1 mark]*
State the intercepts of f.

(b) *[2 marks]*
Find the vertical and horizontal asymptotes of f. Justify your answers.

(c) *[3 marks]*
Given that $f'(x) = \frac{1}{2}e^{x/2}(2 + x)$, find where $f(x)$ is increasing, decreasing and find all relative extrema.

(d) *[3 marks]*
Given that $f''(x) = \frac{1}{4}e^{x/2}(4+x)$, find where $f(x)$ is concave upwards, concave downwards and find all inflection points.

(e) *[4 marks]*
Sketch an accurate graph of $y = f(x)$ on the axes below.



B3. *[10 marks]*

Suppose that a lake is stocked initially with 100 fish and the fish population P thereafter satisfies the differential equation

$$\frac{dP}{dt} = k\sqrt{P},$$

where t is time and k is a constant.

If after 6 months there are 169 fish in the lake, how many will there be after one year?

B4. *[10 marks]*

Find the following integral or show that it diverges:

$$\int_1^\infty \frac{dx}{x(x+1)^2}$$

B5. *[10 marks]*

To fill an order for 500 units of its product, a firm wishes to distribute production between its two plants. The total cost function is given by

$$C(q_1, q_2) = 5q_1^2 + 2q_1q_2 + 2q_2^2$$

where q_1 is the number of units produced in plant 1, and q_2 is the number of units produced in plant 2.

Use the method of Lagrange multipliers to decide how to distribute the order between the plants so as to minimize total cost.

(You may assume the critical point does indeed correspond to minimum cost; but **no marks will be given for any other method than Lagrange multipliers**.)

Solutions to May 2010 Exam, MAT133Y

PART A. MULTIPLE-CHOICE QUESTIONS

A1. ANSWER: Ⓓ.

$$\begin{array}{c} \quad w \quad\; x \quad y \quad\; z \\ \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 8 \\ 2 & -4 & 3 & -1 & 13 \end{array}\right) \end{array} \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 8 \\ 0 & 0 & -1 & 1 & -3 \end{array}\right)$$

$$\xrightarrow{R_2 \to -R_2} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 8 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right)$$

$$w = 2 + 2x - z$$

Alternatively, rather than doing the complete reduction, after the first step, back substitution gives

$$\begin{aligned} y &= 3 + z \\ w &= 8 + 2x - 2y + z = 8 + 2x - 2(3 + z) + z \\ &= 2 + 2x - z \quad \text{as before.} \end{aligned}$$

A2. ANSWER: Ⓑ. Step 1: Find B^{-1}.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\begin{aligned} AB^{-1}A^t &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = (-1) \end{aligned}$$

A3. ANSWER: Ⓔ.

$$3x^2 + y^2 + 2xyy' - 3y^2y' = 0$$

If $x = 1$ and $y = 2$,

$$3 + 4 + 4y' - 12y' = 0$$
$$y' = \frac{7}{8}$$

Equation of the line is

$$\begin{aligned} y - 2 &= \frac{7}{8}(x-1) \\ 8y - 16 &= 7x - 7 \\ 0 &= 7x - 8y + 9 \end{aligned}$$

A4. ANSWER: Ⓐ. Logarithmic differentiation is easiest:

$$\begin{aligned} \ln f &= x + \frac{1}{2}\ln(x+4) - \frac{1}{2}\ln(x+1) \\ \frac{f'}{f} &= 1 + \frac{1}{2(x+4)} - \frac{1}{2(x+1)} \end{aligned}$$

At $x = 0$, $f = e^0\sqrt{\frac{4}{1}} = 2$, so

$$f'(0) = 2\left[1 + \frac{1}{8} - \frac{1}{2}\right] = \frac{5}{4}$$

There are longer ways to do this correctly.

A5. ANSWER: Ⓒ.

$$\begin{aligned} &\lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} && \frac{0}{0} \\ &= \lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + xe^x} && \text{still } \frac{0}{0} \\ &= \lim_{x\to 0} \frac{e^x}{e^x + e^x + xe^x} = \frac{1}{2} \end{aligned}$$

A6. ANSWER: Ⓑ.

$$\begin{aligned} x_{n+1} &= x_n - \frac{4x_n^3 - 36x_n^2 + 77x_n - 40}{12x_n^2 - 72x_n + 77} \\ x_2 &= 1 - \frac{4 - 36 + 77 - 40}{12 - 72 + 77} \\ &= \frac{12}{17} \approx 0.7059 \end{aligned}$$

A7. ANSWER: Ⓒ.

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$$

Only $x = 2$ is in $[0, 3]$.

$$\begin{aligned} f(0) &= 15 \\ f(2) &= 16 - 12 - 24 + 15 = -5 \\ f(3) &= 54 - 27 - 36 + 15 = 6 \end{aligned}$$

So 15 is the maximum and -5 is the minimum.

A8. ANSWER: Ⓒ.

$$\begin{aligned} r(140) - r(60) &= \int_{60}^{140} \frac{dr}{dq} dq \\ &= \int_{60}^{140} \frac{200}{\sqrt{q+4}} dq \\ &= 400\sqrt{q+4}\Big|_{60}^{140} \\ &= 400\left(\sqrt{144} - \sqrt{64}\right) \\ &= 400(12 - 8) = 1600 \end{aligned}$$

A9. ANSWER: Ⓔ.

$$\begin{aligned}&\int_0^{10} 100e^{-0.02t}e^{0.03(10-t)}dt\\&= 100e^{0.3}\int_0^{10} e^{-0.05t}dt\\&= \frac{100}{-0.05}e^{0.3}\left[e^{-0.5}-1\right]\\&= 2000(e^{0.3}-e^{0.2})\\&\approx \$1062\end{aligned}$$

A10. ANSWER: Ⓑ.

$$\frac{1}{e-1}\int_1^e \frac{1}{x}\,dx = \frac{1}{e-1}\ \ln|x|\Big|_1^e = \frac{1}{e-1}$$

A11. ANSWER: Ⓓ.

$$\begin{aligned}u &= x & dv &= e^x\,dx\\ du &= dx & v &= e^x\end{aligned}$$

$$\int_0^1 xe^x dx = xe^x\Big|_0^1 - \int_0^1 e^x dx = (1\cdot e - 0) - (e-1) = 1$$

A12. ANSWER: Ⓔ.

$$\begin{aligned}f(x,y) &= y^2e^{2x} + x^3\ln y - x^3\ln x\\ f_x &= 2y^2e^{2x} + 3x^2\ln y - 3x^2\ln x - \frac{x^3}{x}\\ f_x(1,e) &= 2e^2e^2 + 3\cdot 1\ln e - 3\ln 1 - 1 = 2e^4+2\end{aligned}$$

A13. ANSWER: Ⓓ.

$$\frac{\partial q_A}{\partial p_B} = \frac{-100}{p_A(p_B)^{3/2}} < 0 \text{ and } \frac{\partial q_B}{\partial p_A} = \frac{-100}{p_B(p_A)^{4/3}} < 0$$

A and B are complementary.

(Note that $\dfrac{\partial q_A}{\partial p_A} = \dfrac{-200}{p_A^2\sqrt{p_B}} < 0$ and $\dfrac{\partial q_B}{\partial p_B} = \dfrac{-300}{p_B^2\sqrt[3]{p_A}} < 0$)

A14. ANSWER: Ⓐ.

$$\left.\frac{\partial z}{\partial r}\right|_{(2,3)} = \left.\frac{\partial z}{\partial x}\right|_{(1,2)} \left.\frac{\partial x}{\partial r}\right|_{(2,3)} + \left.\frac{\partial z}{\partial y}\right|_{(1,2)} \left.\frac{\partial y}{\partial r}\right|_{(2,3)} = 4\cdot 3 + 5\cdot(-2) = 2$$

A15. ANSWER: Ⓓ.

$$\begin{aligned} f_x &= 4x - 8 && = 0 \quad \text{when } x = 2 \\ f_y &= 6y - 6y^2 && \\ &= 6y(1-y) && = 0 \quad \text{when } y = 1 \text{ and } y = 0 \end{aligned}$$

The critical points are $(2,0)$ and $(2,1)$.

$$\begin{aligned} f_{xx} &= 4 \\ f_{yy} &= 6 - 12y \\ f_{xy} &= 0 \end{aligned}$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 4(6-12y)$$

$D(2,1) = 4(6-12) = -24 < 0$ so no relative extremum at $(2,1)$.
$D(2,0) = 4(6) = 24 > 0$ and $f_{xx} = 4 > 0$ so relative minimum at $(2,1)$.

PART B. WRITTEN-ANSWER QUESTIONS

B1.

a)

$$\begin{aligned} &V = 100, \quad n = 18, \quad rV = 2, \quad i = 0.025 \\ &P = V(1+i)^{-n} + rVa_{\overline{n}|i} \\ &\quad = 100(1.025)^{-18} + \frac{2\left[1-(1.025)^{-18}\right]}{0.025} \\ &\boxed{P = \$92.82} \end{aligned}$$

b)

$$A = 400,000, \quad n = 12 \times 15 = 180, \quad (1+i)^{12} = (1.035)^2$$

$$400,000 = Ra_{\overline{180}|i}$$

$$R = \frac{400,000i}{1-(1+i)^{-180}} = \frac{400,000\left[(1.035)^{1/6}-1\right]}{1-(1.035)^{-30}}$$

$$\boxed{R = \$3573}$$

c) This is the 120th payment. After the previous payment (the 119th), there were 61 payments remaining, so an outstanding principal of $Ra_{\overline{61}|i}$. The interest on this for the 120th period is

$$iRa_{\overline{61}|i} = R\left[1-(1+i)^{-61}\right] = 3573\left[1-(1.035)^{-61/6}\right] = \boxed{\$1054.51}$$

B2.

a) y intercept: $f(x) = 0$ at $x = 0$. x intercept: $x = 0$ at $y = 0$. So both intercepts are at the origin.

b) $\boxed{\text{V.A.: none}}$ since f is continuous.

H.A.: $\lim_{x\to\infty} xe^{x/2} = \infty$ and

$$\lim_{x\to-\infty} xe^{x/2} = \lim_{x\to-\infty} \frac{x}{e^{-x/2}} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to-\infty} \frac{1}{-\frac{1}{2}e^{-x/2}} = 0$$

$\boxed{\text{H.A.: } y = 0 \text{ at } -\infty}$

c)

	f'	f
$(-\infty,-2)$	$-$	decreasing
$(-2,\infty)$	$+$	increasing

$\boxed{x = -2 \text{ is a minimum}}$ (actually an absolute minimum)

$f(-2) = -\dfrac{2}{e}$

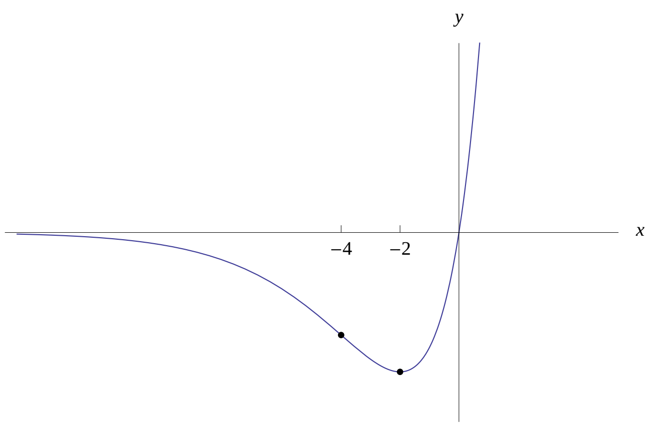
d)

	f''	concavity
$(-\infty, -4)$	$-$	up
$(-4, \infty)$	$+$	down

$x = -4$ is a point of inflection

$f(-4) = -\dfrac{4}{e^2}$

e)



B3.

$$\int \frac{dP}{\sqrt{P}} = \int k\ dt$$
$$2\sqrt{P} = kt + C$$
$$\text{At } t = 0, \quad P = 100$$
$$\text{so } 2\sqrt{100} = C$$
$$\text{and } C = 20.$$
$$2\sqrt{P} = kt + 20$$
$$\text{At } t = \frac{1}{2} \text{ year}, \quad P = 169$$
$$\text{so } 2\sqrt{169} = \frac{k}{2} + 20$$
$$\text{solving, } k = 12.$$
$$2\sqrt{P} = 12t + 20$$
$$\text{At } t = 1, \quad 2\sqrt{P} = 32$$
$$\sqrt{P} = 16$$
$$\boxed{P = 256}$$

Note: t doesn't have to be in years. Say you used months instead. Still $2\sqrt{P} = kt + 20$ just as before. At $t = 6$ months, $2\sqrt{169} = 6k + 20$, so $k = 1$ and $2\sqrt{P} = t + 20$. Then, at $t = 12$, $2\sqrt{P} = 32$, so $\sqrt{P} = 16$ and $\boxed{P = 256}$ as before.

B4.

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
$$A(x+1)^2 + Bx(x+1) + Cx = 1$$

$x = 0$ gives $A = 1$.

$x = -1$ gives $-C = 1$, so $C = -1$.

and anything else, say $x = 1$, gives

$$\begin{aligned} 4A + 2B + C &= 1 \\ 4 + 2B - 1 &= 1 \\ 2B &= -2 \\ \text{and } B &= -1 \end{aligned}$$

$$\begin{aligned} \int_1^\infty \frac{dx}{x(x+1)^2} &= \lim_{R\to\infty} \int_1^R \left[\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right] dx \\ &= \lim_{R\to\infty} \left[\ln|x| - \ln|x+1| + \frac{1}{x+1}\right]_1^R \\ &= \lim_{R\to\infty} \left(\ln R - \ln(R+1) + \frac{1}{R+1}\right) - \left(\ln 1 - \ln 2 + \frac{1}{2}\right) \\ &= \lim_{R\to\infty} \left[\ln\left(\frac{R}{R+1}\right) + \frac{1}{R+1}\right] + \ln 2 - \frac{1}{2} \end{aligned}$$

But $\dfrac{R}{R+1} \to 1$ and $\ln 1 = 0$, also $\dfrac{1}{R+1} \to 0$.

The answer is $\boxed{\ln 2 - \frac{1}{2}} \approx 0.193$.

B5.

$$\begin{aligned} q_1 + q_2 &= 500 \\ L &= 5q_1^2 + 2q_1q_2 + 2q_1^2 - \lambda(q_1 + q_2 - 500) \\ \frac{\partial L}{\partial q_1} &= 10q_1 + 2q_2 - \lambda \\ \frac{\partial L}{\partial q_2} &= 2q_1 + 4q_2 - \lambda \end{aligned}$$

$$\frac{\partial L}{\partial \lambda} = -(q_1 + q_2 - 500)$$

Setting each of these to zero:

$$10q_1 + 2q_1 = \lambda$$
$$2q_1 + 4q_2 = \lambda$$

$$\left(\begin{array}{cc|c} 2 & 4 & \lambda \\ 10 & 2 & \lambda \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & \lambda/2 \\ 0 & -18 & -4\lambda \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & \lambda/18 \\ 0 & 1 & 2\lambda/9 \end{array}\right)$$

Thus, or solving otherwise, $q_1 = \dfrac{\lambda}{18}$ and $q_2 = \dfrac{4\lambda}{18}$

$q_1 + q_2 = 500 \implies \dfrac{5\lambda}{18} = 500 \implies \lambda = 1800$

Hence $\boxed{q_1 = 100 \text{ and } q_2 = 400}$