FACULTY OF ARTS AND SCIENCE

University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2008

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. *[3 marks]*

The system

$$\begin{array}{rcrcrcr} x & - & 2y & - & z & = & 1 \\ -x & + & 2y & + & 5z & = & -1 \\ 2x & - & 4y & - & 6z & = & 2 \end{array}$$

has

Ⓐ a unique solution

Ⓑ no solution

Ⓒ infinitely many solutions with $x = 0$

Ⓓ infinitely many solutions with $y = 0$

Ⓔ infinitely many solutions with $z = 0$

2. *[3 marks]*

$$\lim_{x \to 1} \left(\frac{4x}{x^2 - 1} - \frac{2}{x - 1} \right)$$

Ⓐ $= -2$

Ⓑ $= 1$

Ⓒ $= 2$

Ⓓ $= 0$

Ⓔ does not exist

3. *[3 marks]*

If $f(x) = x^{\sqrt{x}}$, then $f'(4) =$

Ⓐ $8 + 4\ln 4$

Ⓑ $4 + 4\ln 4$

Ⓒ $4 + 8\ln 4$

Ⓓ $8 + 8\ln 4$

Ⓔ $4 - 8\ln 4$

4. *[3 marks]*

If $y^3 + xy = 2x^2$, then when $x = 1$ and $y = 1$, $\dfrac{dy}{dx} =$

Ⓐ $\dfrac{5}{4}$

Ⓑ 1

Ⓒ $\dfrac{1}{2}$

Ⓓ $\dfrac{3}{4}$

Ⓔ $\dfrac{1}{4}$

5. *[3 marks]*

The (absolute) maximum value and (absolute) minimum value of

$$f(x) = \frac{x^2 - 4}{x^2 + 4}$$

on $[-4, 4]$ are

Ⓐ $\max = 1\,, \quad \min = -\dfrac{5}{3}$

Ⓑ No absolute maximum, $\min = -1$

Ⓒ No absolute minimum, $\max = \dfrac{3}{5}$

Ⓓ $\max = \dfrac{3}{5}, \quad \min = -1$

Ⓔ $\max = 1, \quad \min = 0$

6. *[3 marks]*

Let $f(x) = x^{\frac{1}{3}}(4+x)$.

The interval(s) on which f is decreasing is (are)

Ⓐ $(-\infty, -1)$ and $(0, \infty)$

Ⓑ $(-\infty, -1)$

Ⓒ no interval

Ⓓ $(-\infty, 0)$

Ⓔ $(-1, 0)$ and $(0, \infty)$

7. *[3 marks]*

$\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{\frac{x}{3}} =$

Ⓐ ∞

Ⓑ 1

Ⓒ $e^{1/6}$

Ⓓ $e^{2/3}$

Ⓔ $e^{3/2}$

8. *[3 marks]*

$\int_0^1 x^3(1+x^4)^5\, dx =$

Ⓐ $\frac{21}{2}$

Ⓑ $\frac{21}{8}$

Ⓒ $\frac{1}{24}$

Ⓓ $\frac{1}{4}$

Ⓔ $\frac{13}{8}$

9. *[3 marks]*

The area bounded by the y-axis, the straight line $y = ex$ and the curve $y = e^x$ is equal to

Ⓐ $\int_0^1 (e - e^x)\, dx$

Ⓑ $\int_1^e \ln y \, dy$

Ⓒ $\int_0^e (e^x - ex)\, dx$

Ⓓ $\int_1^e (\ln y - \frac{y}{e})\, dy$

Ⓔ $\int_0^1 \frac{y}{e}\, dy + \int_1^e \left(\frac{y}{e} - \ln y\right) dy$

10. *[3 marks]*

The average value of $f(x) = \sqrt{x-4}$ over the interval $[5, 13]$ is

Ⓐ $\frac{13}{4}$

Ⓑ $\frac{13}{6}$

Ⓒ $\frac{52}{3}$

Ⓓ $\frac{1}{6}$

Ⓔ 2

11. *[3 marks]*

$$\int_0^\infty \frac{1}{(x+2)^5}\, dx$$

Ⓐ diverges

Ⓑ $= 0$

Ⓒ $= \frac{1}{16}$

Ⓓ $= -\frac{1}{16}$

Ⓔ $= \frac{1}{64}$

12. *[3 marks]*

If $\dfrac{dy}{dx} = -xy$ and $y = 2$ when $x = 0$ then when $x = 2$, $y =$

Ⓐ 0

Ⓑ 1

Ⓒ e^{-4}

Ⓓ e^{-2}

Ⓔ $2e^{-2}$

13. *[3 marks]*

If $e^{x^2} = z \ln y$ then at the point $(1, e, e)$, $\dfrac{\partial z}{\partial y} =$

Ⓐ $3e$

Ⓑ $2e$

Ⓒ e

Ⓓ $2 - e$

Ⓔ -1

14. *[3 marks]*

Let $z = \ln(x^2 + y^2)$ where

$$x = 2s - 3t$$
$$y = 4s + t^2.$$

$\dfrac{\partial z}{\partial s} =$

Ⓐ $\dfrac{xs + ty}{x^2 + y^2}$

Ⓑ $\dfrac{2t + 3s}{(2s - 3t)^2 + (t^2 + 4s)^2}$

Ⓒ $\dfrac{4x + 8y}{x^2 + y^2}$

Ⓓ $\dfrac{2t + 3st^2}{(2s - 3t)^2 + (t^2 + 4s)^2}$

Ⓔ $xs \ln(x^2 + y^2)$

15. *[3 marks]*

$$\int_1^3 \int_{-2}^1 (4x^3 + 6xy^2)\,dy\,dx =$$

Ⓐ 312

Ⓑ 280

Ⓒ 292

Ⓓ 336

Ⓔ 372

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[11 marks]*

Evaluate $\displaystyle\int_2^3 \frac{x+1}{x(x-1)^2}\,dx$

B2. *[11 marks]*

Over the next 7 years the profits of a business at time t (in years) are estimated to be $40,000t$ dollars per year. The business is to be sold at a price equal to the present value of these future profits. At what price should the business be sold if interest is compounded continuously at 8%?

B3. *[11 marks]*

The production function for a firm is

$$P(l,k) = 18l + 20k - 2l^2 - 4k^2 - lk$$

where l is the number of labour-hours per week and k is the capital (in thousands of dollars per week). Find values of l and k to maximize production (Use the second derivative test to verify that you have at least a relative maximum).

B4. *[11 marks]*

Use the method of Lagrange multipliers to find the critical points of $3x + y + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

B5. *[11 marks]*

An $8,000 loan is to be repaid with semiannual payments of $500 each until outstanding principal is less than $500 (just after the last full $500 payment). Interest is 6% compounded semiannually and the first $500 payment is to be made six months after the loan is taken out.

(a) *[6 marks]*
How many full $500 payments will be made while repaying the loan?

(b) *[5 marks]*
Six months after the last full $500 payment, a smaller final payment will be made to repay the remaining outstanding principal plus interest. What will be the amount of the final payment?

Solutions to April 2008 Exam, MAT133Y

PART A

1. ANSWER: Ⓔ

$$\begin{pmatrix} 1 & -2 & -1 & | & 1 \\ -1 & 2 & 5 & | & -1 \\ 2 & -4 & -6 & | & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & -1 & | & 1 \\ 0 & 0 & 4 & | & 0 \\ 0 & 0 & -4 & | & 0 \end{pmatrix}$$

$$\boxed{z = 0}$$

$$x - 2y - z = 0 \Rightarrow \boxed{x = 2y}$$

2. ANSWER: Ⓑ

$$\begin{aligned} &= \lim_{x\to 1} \frac{4x - 2(x+1)}{x^2 - 1} \\ &= \lim_{x\to 1} \frac{2x - 2}{x^2 - 1} \\ &= \lim_{x\to 1} \frac{2}{x+1} = \boxed{1} \end{aligned}$$

3. ANSWER: Ⓐ

$$\begin{aligned} \ln f &= \sqrt{x}\ln x \\ \frac{1}{f}f' &= \frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x} \\ &= \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \\ &= \frac{\ln x + 2}{2\sqrt{x}} \\ \frac{1}{f(4)}f'(4) &= \frac{2 + \ln 4}{2 \cdot 2} \end{aligned}$$

$$\text{but } f(4) = 4^{\sqrt{4}} = 16 \qquad \text{so } \boxed{f'(4) = 8 + 4\ln 4}$$

4. ANSWER: Ⓓ

$$3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 4x$$

At $(1,1)$

$$3\frac{dy}{dx} + 1 + \frac{dy}{dx} = 4$$

$$4\frac{dy}{dx} = 3$$

$$\boxed{\frac{dy}{dx} = \frac{3}{4}}$$

5. ANSWER: Ⓓ

$$f(x) = \frac{x^2-4}{x^2+4} = \frac{x^2+4-8}{x^2+4} = 1 - \frac{8}{x^2+4}$$

$$f'(x) = \frac{16x}{(x^2+4)^2}$$

crit pt at $x = 0$ only

$$f(0) = -1$$

$$f(-4) = f(4) = \frac{12}{20} = \frac{3}{5}$$

Cont. fcu. on closed interval must have minimum and maximum. Only chances at $x = 0,\ -4,\ 4$. So $\boxed{\max = \frac{3}{5},\ \min = -1}$

6. ANSWER: Ⓑ

$$f(x) = 4x^{\frac{1}{3}} + x^{\frac{4}{3}}$$

$$f'(x) = \frac{4}{3}x^{-\frac{2}{3}} + \frac{4}{3}x^{\frac{1}{3}}$$

$$f'(x) = \frac{4(1+x)}{3x^{\frac{2}{3}}}$$

$x^{\frac{2}{3}} > 0$ always, so $f' < 0$ for $x < -1$ only.
f decreasing on $(-\infty, -1)$

7. ANSWER: Ⓓ

$$y = \left(1 + \frac{2}{x}\right)^{\frac{x}{3}}$$

$$\ln y = \frac{x}{3} \ln\left(1 + \frac{2}{x}\right)$$

$$= \frac{\ln\left(1 + \frac{2}{x}\right)}{\frac{3}{x}} \qquad \frac{0}{0}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}} \quad \frac{-2}{x^2}}{-\frac{3}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2}{3} \cdot \frac{1}{1 + \frac{2}{x}} = \frac{2}{3}$$

$$y = e^{\ln y} \Rightarrow \boxed{e^{\frac{2}{3}}}$$

8. ANSWER: Ⓑ

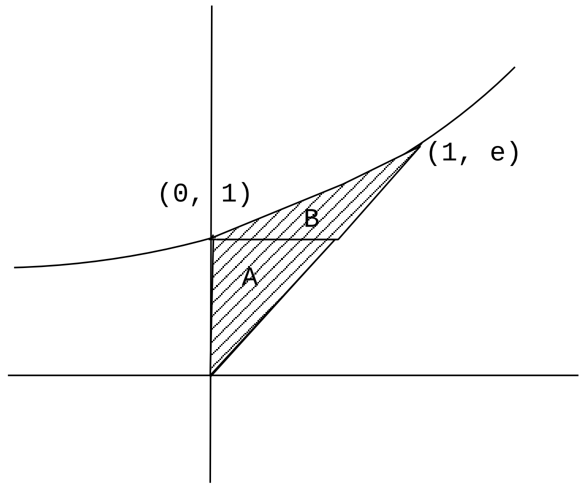
$$u = 1 + x^4 \qquad du = 4x^3\ dx$$

$$\int = \int_1^2 \frac{u^5}{4}\ du$$

$$= \frac{u^6}{24}\bigg|_1^2 = \frac{64 - 1}{24} = \frac{63}{24}$$

$$= \boxed{\frac{21}{8}}$$

9. ANSWER: Ⓔ



$$e^x = ex \text{ at } x = 1$$
$$y = e$$
$$x = \frac{y}{e} \qquad x = \ln y$$

are the two curves

$$\text{Area } A = \int_0^1 \frac{y}{e}\, dy$$
$$\text{Area } B = \int_1^e \left(\frac{y}{e} - \ln y\right) dy$$

Note: C would be correct if it said $\int_0^1$.

10. ANSWER: Ⓑ

$$\begin{aligned}\bar{f} &= \frac{1}{8}\int_5^{13} \sqrt{x-4}\, dx \\ &= \frac{1}{8}(x-4)^{3/2} \cdot \frac{2}{3}\Big|_5^{13} \\ &= \frac{1}{12}\left(9^{3/2} - 1^{3/2}\right) \\ &= \frac{1}{12}(27-1) = \boxed{\frac{13}{6}}\end{aligned}$$

11. ANSWER: Ⓔ

$$\begin{aligned}\int_0^\infty \frac{1}{(x+2)^5}\,dx &= \lim_{R\to\infty}\int_0^R \frac{1}{(x+2)^5}\;dx\\ &= \lim_{R\to\infty} -\frac{1}{(x+2)^4\cdot 4}\Big|_0^R\\ &= \lim_{R\to\infty} -\frac{1}{4(R+2)^4}+\frac{1}{4\cdot 2^4}\\ &= \boxed{\frac{1}{64}}\end{aligned}$$

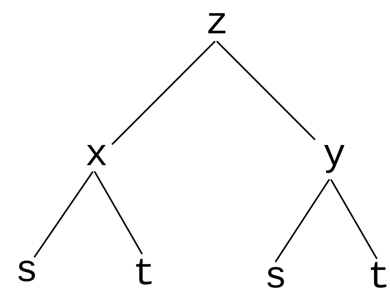
12. ANSWER: Ⓔ

$$\begin{aligned}\frac{dy}{y} &= -x\;dx\\ \ln y &= -\frac{x^2}{2}+C \qquad (y>0)\\ y &= Ae^{-\frac{x^2}{2}}\\ 2 &= Ae^0\\ y &= 2e^{-\frac{x^2}{2}}\\ \text{at } x &= 2\\ y &= 2e^{-4/2} = \boxed{2e^{-2}}\end{aligned}$$

13. ANSWER: Ⓔ

$$\begin{aligned}0 &= \frac{\partial z}{\partial y}\ln y+\frac{z}{y}\\ \frac{\partial z}{\partial y} &= -\frac{z}{y\ln y}\\ &= -\frac{e}{e\ln e} = \boxed{-1}\end{aligned}$$

14. ANSWER: Ⓒ



$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= \frac{2x}{x^2+y^2}\cdot 2 + \frac{2y}{x^2+y^2}\cdot 4 \\ &= \boxed{\frac{4x+8y}{x^2+y^2}}\end{aligned}$$

15. ANSWER: Ⓐ

$$\begin{aligned}\int_1^3\int_{-2}^1 (4x^3+6xy^2)\,dy\,dx &= \int_1^3 \Big[4x^3y+2xy^3\Big]_{y=-2}^1\,dx \\ &= \int_1^3 \big[(4x^3+2x)-(-8x^3-16x)\big]\,dx \\ &= \int_1^3 (12x^3+18x)\,dx \\ &= \Big[3x^4+9x^2\Big]_1^3 \\ &= (3\cdot 81+81)-(3+9) \\ &= 324-12 \\ &= \boxed{312}\end{aligned}$$

PART B

B1.

$$\frac{x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$A(x-1)^2 + Bx(x-1) + Cx = x+1$$

$$x = 1 \qquad C = 2$$

$$x = 0 \qquad A = 1$$

$$x = 2 \text{ for example} \qquad A + 2B + 2C = 3$$

$$1 + 2B + 4 = 3$$

$$2B = -2$$

$$B = -1$$

$$\begin{aligned}\int_2^3 \frac{x+1}{x(x-1)^2}\,dx &= \int_2^3 \Big[\frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2}\Big]\,dx \\ &= \Big[\ln|x| - \ln|x-1| - \frac{2}{x-1}\Big]_2^3 \\ &= \ln\frac{3}{2} - \ln 2 - (1-2) \\ &= \boxed{1 + \ln\frac{3}{4}} \\ &\approx \boxed{.712}\end{aligned}$$

B2.

$$\text{P.V.} = \int_0^7 40,000te^{-.08t}dt$$

$$u = t \qquad dv = e^{-.08t}$$

$$du = dt \qquad v = -\frac{-e^{-.08t}}{.08}$$

$$\begin{aligned}&= 40,000\Big[\frac{-te^{-.08t}}{.08}\Big|_0^7 + \frac{1}{.08}\int_0^7 e^{-.08t}dt\Big] \\ &= 40,000\Big[\frac{-7}{.08}e^{-.56} - \frac{1}{(.08)^2}e^{-.08t}\ \Big|_0^7\Big] \\ &= 500,000\Big[-7e^{-.56} - \frac{1}{(.08)}e^{-.56} + \frac{1}{0.8}\Big] \\ &= \boxed{\$680,711.63}\end{aligned}$$

B3.

$$\frac{\partial P}{\partial l} = 18 - 4l - k = 0$$
$$\frac{\partial P}{\partial k} = 20 - 8k - l = 0$$
$$4l + k = 18$$
$$l + 8k = 20$$
$$31l = 124 \Rightarrow l = 4 \Rightarrow k = 2$$

So only critical value is $\boxed{l = 4,\ k = 2}$

$$\frac{\partial^2 P}{\partial l^2} = -4 \qquad \frac{\partial^2 P}{\partial k^2} = -8$$
$$\frac{\partial^2 P}{\partial l \partial k} = -1$$

$$D = P_{ll}P_{kk} - (P_{lk})^2 = 32 - 1 = 31 > 0$$

The critical point **is** a local extremum and since P_{ll} (for example) $= -4 < 0$ this crit pt is a local maximum.

B4.

$$\mathcal{L} = 3x + y + z - \lambda(x^2 + y^2 + z^2 - 1)$$

$$\mathcal{L}_x = 3 - 2\lambda x = 0$$
$$\mathcal{L}_y = 1 - 2\lambda y = 0 \qquad \text{at critical point(s)}$$
$$\mathcal{L}_z = 1 - 2\lambda z = 0$$

$$x = \frac{3}{2\lambda} \qquad y = \frac{1}{2\lambda} \qquad z = \frac{1}{2\lambda}$$

$$\text{and } \mathcal{L}_\lambda = 0 \Rightarrow x^2 + y^2 + z^2 = 1$$
$$\text{so } \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$4\lambda^2 = 11 \qquad 2\lambda = \pm\sqrt{11}$$

$$\boxed{(x, y, z) = (\tfrac{3}{\sqrt{11}},\ \tfrac{1}{\sqrt{11}},\ \tfrac{1}{\sqrt{11}})} \text{ when } 2\lambda = +\sqrt{11}$$

$$\boxed{(x, y, z) = (-\tfrac{3}{\sqrt{11}},\ -\tfrac{1}{\sqrt{11}},\ -\tfrac{1}{\sqrt{11}})} \text{ when } 2\lambda = -\sqrt{11}$$

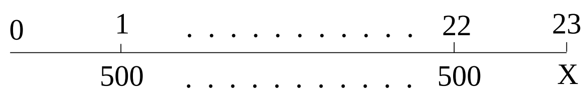
B5.

(a)

$$8000 = 500a_{\overline{n}|.03}$$
$$16 = \frac{1-(1.03)^{-n}}{.03}$$
$$(1.03)^{-n} = 1 - .48 = .52$$
$$-n \ \ln(1.30) = \ln .52$$
$$n = -\frac{\ln .52}{\ln(1.03)} \approx 22.12\ldots$$

$$\boxed{22 \ \text{full} \ \text{payments}}$$

(b)



Let x be the last payment

$$8000 = 500a_{\overline{22}|.03} + x(1.03)^{-23}$$
$$= (1.03)^{23}\left[8000 - 500\left(\frac{1-(1.03)^{-22}}{.03}\right)\right]$$

$$\boxed{x = \$62.25}$$