FACULTY OF ARTS AND SCIENCE University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2007

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. [3 marks]

The system of equations

has

- (A) no solutions
- (B) a unique solution
- © infinitely many solutions with one parameter
- (D) infinitely many solutions with two parameters
- (E) infinitely many solutions with three parameters
- 2. [3 marks]

Let
$$h(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{for } x \neq 1\\ a - 2 & \text{for } x = 1 \end{cases}$$

Then h(x) is continuous everywhere for a equal to

(A) 4
(B) 3
(C) 1
(D) 2
(E) -2

4. [3 marks]

The slope of the tangent line to the curve $x^3 + 3y^2 = 4$ at (1,1) is

- (A) undefined
- $\begin{array}{c} \textcircled{B} & -\frac{1}{3} \\ \hline \textcircled{C} & \frac{4}{3} \\ \hline \textcircled{D} & \frac{3}{4} \\ \hline \textcircled{E} & -\frac{1}{2} \end{array}$

On the interval
$$(\frac{1}{2}, 2)$$
, the function $f(x) = e^{-(x^2 + x^{-2})}$ has
(A) a local minimum but no local maximum
(B) a local maximum but no local minimum

- \bigcirc ~ a local maximum and a local minimum
- (D) neither a local maximum nor a local minimum
- (c) two local minima and no local maximum

6. [3 marks]

$$\lim_{x \to \infty} (x^2 + 1)^{\frac{1}{x^2 + 2}}$$
(A) does not exist
(B) $= e^{\frac{1}{2}}$
(C) $= e$
(D) $= 0$
(E) $= 1$

7. [3 marks]

When using the Trapezoidal Rule with the interval [-2, 6] divided into n = 4 subintervals the approximate value of

$$\int_{-2}^{6} \sqrt{1+x^2} \, dx$$

is closest to

- **A** 23.04
- **B** 21.55
- © 22.19
- **D** 15.36
- **E** 14.37

$$\int_{-1}^{1} \frac{x^{\frac{2}{3}} dx}{(2+x^{\frac{5}{3}})^{3}} =$$
(A) $-\frac{1}{3}$
(B) $\frac{3}{5} \ln 27$
(C) $-\frac{64}{27}$
(D) $\frac{1}{5}$
(E) $\frac{4}{15}$

9. [3 marks]

$$\int_{1}^{e} x \ln x \, dx =$$
(A) $\frac{e^2 - e + 1}{2}$
(B) $\frac{1}{2}$
(C) $\frac{e^2}{2}$
(D) $\frac{e^2 + 1}{4}$
(E) 0

10. [3 marks]

$$\int \frac{3x-4}{(x-1)(x-2)} dx =$$
(A) $3\ln|(x-1)(x-2)| + C$
(B) $3\ln\left|\frac{x-2}{x-1}\right| + C$
(C) $3\ln\left|\frac{x-1}{x-2}\right| + C$
(D) $\ln\left|(x-1)(x-2)^2\right| + C$
(E) $\ln\left|\frac{x-1}{(x-2)^2}\right| + C$

$$\int_8^\infty \frac{1}{\sqrt[3]{x}} \, dx$$

- (A) diverges
- B = 6
- $\bigcirc = -6$
- D = 4
- E = -4

12. [3 marks] If $f(x, y, z) = \frac{x^2 y^3}{z^4}$, $f_{xyz}(2, -1, -2) =$ (A) $-\frac{3}{8}$ (B) $-\frac{3}{2}$ (C) $-\frac{1}{4}$ (D) $\frac{3}{2}$ (E) $\frac{3}{8}$

13. [3 marks]

If two products called A and B have the joint demand functions

$$q_A(p_A, p_B) = 1000 - 24p_A + 2p_A^2 - 40p_B + p_B^2$$

and

$$q_B(p_A, p_B) = 2000 + 60p_A - 5p_A^2 - 36p_B - 3p_B^2$$

then the products are competitive provided

(A)
$$p_A < 6$$
 and $p_B > 20$
(B) $p_A > 6$ and $p_B > 20$
(C) $p_A < 6$ and $p_B < 20$
(D) $(p_A < 6$ and $p_B < 20)$ or $(p_A > 6$ and $p_B > 20)$
(E) $p_A > 6$ and $p_B < 20$

If x(x+y+z) = yz then when (x, y, z) = (1, 2, 3), $\frac{\partial x}{\partial z} =$

15. [3 marks]

If the joint demand functions for the products A and B are given by

$$q_A = \frac{4}{p_A \sqrt[3]{p_B}} \quad q_B = \frac{6}{p_B \sqrt{p_A}}$$

and the joint cost function is given by

$$C = q_A{}^2 + 2q_B$$

 $\frac{\partial C}{\partial p_A} =$

then when $p_A = 4$ and $p_B = 1$,

- (A) $-\frac{11}{4}$ (B) $-\frac{5}{4}$ (C) 2 (D) -4
- E $\frac{1}{4}$

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[9 marks]*

Tom owes Jerry two debts:

- \$1000 due now; and
- \$3000 plus interest at 5% compounded quarterly, due in 3 years.

They have agreed that the debts will be repaid in two payments:

— the first payment to be made in 4 years;

— the second payment to be one-third the amount of the first, to be made in 5 years.

What should the amounts of the first and second payments be, if money is worth 6% compounded semi-annually?

B2. [11 marks]

Find the values of x and y which minimize the function f(x,y) = xy subject to the constraint $\frac{1}{x} + \frac{1}{y} = 2$ by using Lagrange Multipliers (no need to verify your answer is a minimum and no marks will be given for any other method).

B3. [11 marks]

A bakery produces oatmeal and chocolate chip cookies. It costs 1/kg to make oatmeal cookies and 2/kg to make chocolate chip cookies. If the bakery sells q_o kg of oatmeal cookies for p_o/kg and q_c kg of chocolate chip cookies for p_c/kg , then the joint demand functions are given by:

$$q_o = 100(p_c - p_o)$$
 $q_c = 500 + 100(p_o - 2p_c)$

Find the prices p_o and p_c that result in maximum profit for the bakery.

(Be sure to verify that you actually get at least a local maximum by using the second derivative test for functions of **two** variables.)

B4. [12 marks]

Solve the following problems showing all your work.

- (a) [6 marks] If $\frac{dy}{dx} = xe^y$ and y = 0 when x = 0, what is y when x = 1?
- (b) [6 marks] If $\frac{dN}{dt} = N(1-N)$ and $N = \frac{1}{2}$ when t = 1, what is N when t = 2? [You may assume that N(t) is always between 0 and 1.]

B5. [12 marks]

In both (a) and (b), your final answer should be a number rounded to two decimal places.

(a) [6 marks]

Find the present value of a 2 year continuous annuity at an annual rate of 6% compounded continuously if the rate of payment at time t is (5-t) million dollars per year.

(b) [6 marks] Find $\int_0^1 \int_1^2 (2ye^x - 5xe^y) \, dx \, dy$.

Solutions to April 2007 Exam, MAT133Y PART A

1. ANSWER: D

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & | & 2 \\ 0 & 1 & 1 & 2 & 0 & | & 2 \\ 0 & 0 & 1 & 1 & 2 & | & 3 \end{pmatrix}$$

is already in row echelon form.

#var - #non - zero rows = # parameters 5 - 3 = 2

2. ANSWER: A

To be cont.,

$$\lim_{x \to 1} h(x) = h(1) = a - 2$$
$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$
so $a - 2 = 2$
$$a = 4$$

3. ANSWER: **(E)**

$$\ln f = \left(\frac{1}{x} + \ln x\right) \ln(\sqrt{x} + 1)$$
$$\frac{1}{f}f' = \left(-\frac{1}{x^2} + \frac{1}{x}\right) \ln(\sqrt{x} + 1) + \left(\frac{1}{x} + \ln x\right) \frac{1}{\sqrt{x} + 1} \cdot \frac{1}{2\sqrt{x}}$$

multiplying these by $f = (\sqrt{x} + 1)^{\frac{1}{x} + \ln x}$, gets E

4. ANSWER: **E**

$$\begin{aligned} 3x^2 + 6yy' &= 0\\ y' &= -\frac{x^2}{2y}\\ &= -\frac{1}{2} \quad \text{at} \quad (1,1) \end{aligned}$$

5. ANSWER: ^(B)

$$f'(x) = -e^{-(x^2 + x^{-2})} \left[2x - \frac{2}{x^3} \right] = \frac{-2e^{-(x^2 + x^{-2})}}{x^3} (x^4 - 1)$$

$$f'(x) = 0 \text{ only at } x = 1,$$

the only crit pt., on $(\frac{1}{2}, 2).$

$$\frac{1}{2} < x < 1 \Rightarrow f' > 0$$

$$(1,2) \Rightarrow f' < 0$$
local max at $x = 1$

and that's all, so $\ \textcircled{B}$

6. ANSWER: (E)

$$y = (x^2 + 1)^{\frac{1}{x^2 + 2}}$$
$$\ln y = \frac{\ln(x^2 + 1)}{x^2 + 2} \quad \frac{\infty}{\infty} \text{ as } x \to \infty$$
$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\frac{2x}{x^2 + 1}}{2x} = \lim_{x \to \infty} \frac{1}{x^2 + 1}$$
$$= 0$$
$$\ln y \to 0$$
$$y = e^{\ln y} \to e^0 = 1$$

7. ANSWER: (A)

$$x_0 = -2$$
$$x_1 = 0$$
$$x_2 = 2$$
$$x_3 = 4$$
$$x_4 = 6$$

$$\Delta_x = \frac{6 - (-2)}{4} = 2$$

$$T_4 = \frac{\Delta_x}{2} [y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= 1[\sqrt{1+4} + 2\sqrt{1+0} + 2\sqrt{1+4} + 2\sqrt{1+16} + \sqrt{1+36}]$$

$$\approx 23.037 \text{ so } \textbf{A}$$

8. ANSWER: E

Let

$$u = 2 + x^{\frac{5}{3}} \qquad x = -1 \implies u = 1 \qquad x = 1 \implies u = 3$$
$$du = \frac{5}{3}x^{\frac{2}{3}} dx$$
$$\text{Integral} = \frac{3}{5} \int^{3} \frac{du}{3}$$

$$\begin{aligned} \text{ntegral} &= \frac{3}{5} \int_{1}^{5} \frac{du}{u^{3}} \\ &= \frac{\frac{3}{5}}{-2} u^{-2} \Big|_{1}^{3} \\ &= -\frac{3}{10} \Big[\frac{1}{9} - 1 \Big] = \frac{3}{10} \cdot \frac{8}{9} = \frac{4}{15} \end{aligned}$$

9. ANSWER: D

$$u = \ln x \qquad dv = x \, dx du = \frac{1}{x} \, dx \qquad v = \frac{x^2}{2}$$

Integ. = $\frac{x^2}{2} \ln x \Big|_1^e - \frac{1}{2} \int_1^e x \, dx = \frac{e^2}{2} - \frac{1}{2} \frac{x^2}{2} \Big|_1^e = \frac{e^2}{2} - \frac{1}{4} (e^2 - 1) = \frac{1}{4} e^2 + \frac{1}{4}$

10. ANSWER: **()**

$$\frac{3x-4}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$A(x-2) + B(x-1) = 3x - 4$$

$$x = 1: -A = -1 \text{ so } A = 1$$

$$x = 2: \quad B = 2$$

Integral =
$$\int \left[\frac{1}{x-1} + \frac{2}{x-2}\right] dx$$

= $\ln |x-1| + 2\ln |x-2| + C$
= $\ln(|x-1||x-2|^2) + C$

11. ANSWER: (A)

$$\lim_{R \to \infty} \int_8^R x^{-\frac{1}{3}} dx$$
$$= \lim_{R \to \infty} \frac{3}{2} x^{\frac{2}{3}} \Big|_8^R$$
$$= \lim_{R \to \infty} \frac{3}{2} [R^{\frac{2}{3}} - 4] \to \infty, \quad \text{so } (\mathbb{A})$$

12. ANSWER: ①

$$f_x = \frac{2xy^3}{z^4}$$

$$f_{xy} = \frac{6xy^2}{z^4}$$

$$f_{xyz} = -\frac{24xy^2}{z^5}$$

$$f_{xyz}(2, -1, -2) = -\frac{24 \cdot 2 \cdot (-1)^2}{(-2)^5}$$

$$= \frac{48}{32} = \frac{3}{2}$$

13. ANSWER: (A)

$$\frac{\partial q_A}{\partial p_B} = -40 + 2p_B > 0 \Leftrightarrow p_B > 20$$
$$\frac{\partial q_B}{\partial p_A} = 60 - 10p_A > 0 \Leftrightarrow p_A < 6$$
so $p_B > 20$ and $p_A < 6$

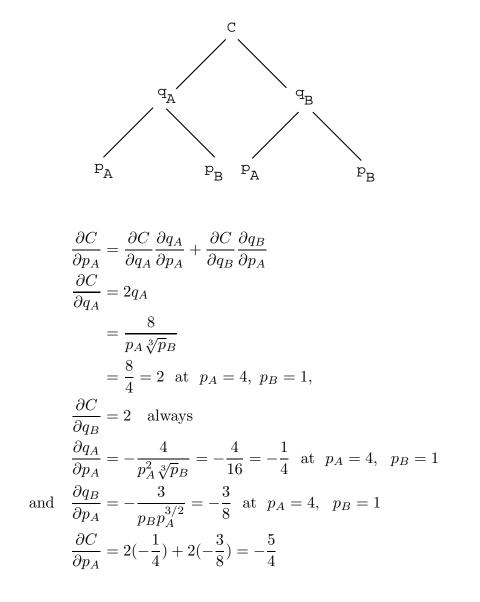
14. ANSWER: ©

$$\frac{\partial x}{\partial z}(x+y+z) + x(\frac{\partial x}{\partial z}+1) = y$$

at (1,2,3)
$$\frac{\partial x}{\partial z} \cdot 6 + \frac{\partial x}{\partial z} + 1 = 2$$

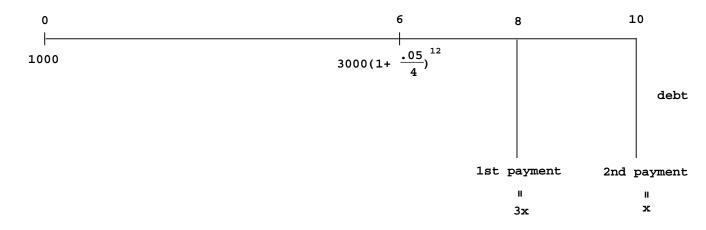
$$7\frac{\partial x}{\partial z} = 1$$

$$\frac{\partial x}{\partial z} = \frac{1}{7}$$









We show the calculation at the end of the 5th yr (for convenience only):

$$1000(1.03)^{10} + 3000\left(1 + \frac{.05}{4}\right)^{12}(1.03)^4 = 3x(1.03)^2 + x$$
$$x = \frac{1000(1.03)^{10} + 3000\left(1 + \frac{0.5}{4}\right)^{12}(1.03)^4}{1 + 3(1.03)^2}$$
$$x = 1258.33$$
$$3x = 3775.00$$
First payment = \$3775.00

2nd payment = \$1258.33

B2.

$$\mathcal{L} = xy - \lambda \left(\frac{1}{x} + \frac{1}{y} - 2\right)$$
$$\mathcal{L}_x = y + \frac{\lambda}{x^2} = 0 \quad \to x^2 y = -\lambda$$
$$\mathcal{L}_y = x + \frac{\lambda}{y^2} = 0 \quad \to xy^2 = -\lambda$$

now divide the 2nd equation by the 1st :

$$\frac{x^2y}{xy^2} = \frac{-\lambda}{-\lambda}$$
$$\frac{x}{y} = 1 \text{ and so } x = y$$

(The division is OK because x = 0 and/or y = 0 is forbidden by the constraint $\frac{1}{x} + \frac{1}{y} = 2$, otherwise known as)

$$\mathcal{L}_{\lambda} = 0$$

But if $x = y$, $\frac{1}{x} + \frac{1}{x} = 2$ and $x = 1$, so $y = 1$

Note that there is a value of λ , namely $\lambda = -1$, but we don't really care about this.

B3. Let profit = π Cost = $q_0 + 2q_c$ Revenue = $p_0q_0 + p_cq_c$ π = Revenue - Cost $\pi = q_0p_0 + q_cp_c - q_0 - 2q_c = q_0(p_0 - 1) + q_c(p_c - 2)$ $\pi = 100(p_c - p_0)(p_0 - 1) + [500 + 100(p_0 - 2p_c)](p_c - 2)$ $\frac{\partial \pi}{\partial p_c} = 100[(p_0 - 1) - 2(p_c - 2) + (5 + p_0 - 2p_c)] = 100[2p_0 - 4p_c + 8] = 0$ $\frac{\partial \pi}{\partial p_0} = 100[-(p_0 - 1) + (p_c - p_0) + (p_c - 2)] = 100[-2p_0 + 2p_c - 1]$ Adding the 2 equations gives $100[-2p_c + 7] = 0$ so $p_c = 3.5$ but then using $\frac{\partial \pi}{\partial p_0} = 0$, $-2p_0 + 7 - 1 = 0$, so $p_0 = 3$. The only critical point is $p_0 = \$3.00$ and $p_c = \$3.50$

cal point is
$$p_0 = \$3.00$$
 and $p_c = \$3.50$
 $\frac{\partial^2 \pi}{\partial p_c^2} = -400$
 $\frac{\partial^2 \pi}{\partial p_0^2} = -200$
 $\frac{\partial^2 \pi}{\partial p_0 \partial p_c} = 200$ always
 $D = \frac{\partial^2 \pi}{\partial p_c^2} \frac{\partial^2 \pi}{\partial p_c^2} - \left(\frac{\partial^2 \pi}{\partial p_0 \partial p_c}\right)^2 = (-400)(-200) - (200)^2$
 $= 40,000 > 0$

so our crit. pt. is a local extremum and $\frac{\partial^2 \pi}{\partial p_c^2} < 0$ so it is a local max.

B4.

(a)

$$\int e^{-y} dy = \int x dx$$

 $-e^{-y} = \frac{x^2}{2} + C.$ At $x = 0, y = 0$, this says
 $-e^0 = 0 + C$ so $C = -1$
 $-e^{-y} = \frac{x^2}{2} - 1$
When $x = 1, -e^{-y} = \frac{1}{2} - 1 = -\frac{1}{2}$
 $e^{-y} = \frac{1}{2}$
 $e^y = 2$
 $y = \ln 2 \approx .693$

(b)

$$\int \frac{dN}{N(1-N)} = \int dt$$

$$\int \left(\frac{1}{N} + \frac{1}{1-N}\right) dN = t + C$$

$$\ln N - \ln(1-N) = t + C. \quad \text{At} \quad t = 1, N = \frac{1}{2}, \text{ this says}$$

$$\ln \frac{1}{2} - \ln(\frac{1}{2}) = 1 + C$$

$$C + 1 = 0 \quad \text{and} \quad C = -1$$
Hence,
$$\ln \frac{N}{1-N} = t - 1$$
When
$$t = 2, \ln \frac{N}{1-N} = 1$$

$$\frac{N}{1-N} = e$$

$$N = e - eN$$

$$N(1+e) = e$$

$$\boxed{N = \frac{e}{1+e}} \approx .731$$

B5.

(a)

P.V. =
$$\int_{0}^{2} (5-t)e^{-.06t} dt$$

 $u = 5-t$ $dv = e^{-.06t} dt$
 $du = -dt$ $v = -\frac{e^{-.06t}}{.06} dt$
 $= -\frac{(5-t)}{.06}e^{-.06t}\Big|_{0}^{2} - \frac{1}{.06}\int_{0}^{2}e^{-.06t} dt$
 $= \frac{-3e^{-.12}+5}{.06} + \frac{1}{(.06)^{2}}e^{-.06t}\Big|_{0}^{2}$
 $= \frac{5-3e^{-.12}}{.06} + \frac{e^{-.12}-1}{(.06)^{2}} \approx 7.576 \cdots$
so 7.58 million dollars.

(b)

$$\begin{split} &= \int_0^1 \left[2ye^x - \frac{5x^2}{2}e^y \right]_{x=1}^{x=2} dy \\ &= \int_0^1 \left[(2e^2y - 10e^y) - (2ey - \frac{5}{2}e^y) \right] dy \\ &= \int_0^1 \left[(2e^2 - 2e)y - \frac{15}{2}e^y \right] dy \\ &= \left[(2e^2 - 2e)\frac{y^2}{2} - \frac{15}{2}e^y \right]_0^1 \\ &= (e^2 - e - \frac{15}{2}e) - (-\frac{15}{2}) \\ &= \frac{15}{2} + e^2 - \frac{17e}{2} \\ &= -8.216 \\ &\text{so} \quad \boxed{-8.22} \end{split}$$