FACULTY OF ARTS AND SCIENCE

University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2004

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. *[3 marks]*

   If $f(x) = \dfrac{(x^2 + e^{2x})^3 e^{-2x}}{(1 + x - x^2)^{2/3}}$, then $f'(0) =$

   A. 0

   B. 1

   C. $e^{\frac{10}{3}}$

   D. $\dfrac{10}{3}$

   E. $e^{\frac{16}{3}}$

2. *[3 marks]*

   Let $f(x) = \begin{cases} e^{-x} & x \leq 0 \\ ax + b & x > 0 \end{cases}$ where a and b are constants.

   If f is differentiable at $x = 0$, then $f(x) = 0$ when $x =$

   A. 1

   B. 2

   C. $\ln 2$

   D. $\ln 3$

   E. e

3. *[3 marks]*

The graph of $y = (1 - \ln x)^2$ has

A. no inflection points

B. an inflection point at $x = 1$

C. an inflection point at $x = 2$

D. an inflection point at $x = \sqrt{e}$

E. an inflection point at $x = e^2$

4. *[3 marks]*

The largest value attained by $y = x \ln x$ if $0 < x \leq e$ is

A. e

B. $\frac{1}{e}$

C. 0

D. 1

E. 2

5. *[3 marks]*

$\lim_{x \to 1} x^{\frac{1}{1-x}} =$

A. e

B. $\frac{1}{e}$

C. 1

D. 0

E. undefined

6. *[3 marks]*

If $F(x) = \int_x^7 e^t \ln |t| \, dt$, then $F''(1) =$

A. $-e$

B. e

C. 0

D. $-2e$

E. $2e$

7. *[3 marks]*

$$\int_0^1 \frac{x}{x^2+1}\,dx =$$

A. $= \ln\sqrt{2}$

B. diverges

C. $= e$

D. $= 0$

E. $= \ln 2 - 1$

8. *[3 marks]*

The area between $f(x) = \dfrac{e^x}{1+e^x}$ and the x-axis from $x = 0$ to $x = \ln 5$ is

A. $\ln 3$

B. $\dfrac{1}{3}$

C. $\ln 4$

D. $\dfrac{4}{\ln 5}$

E. $\dfrac{5}{6}$

9. *[3 marks]*

The area bounded by the graphs of $y = x^{\frac{2}{3}}$ and $y = x^{\frac{3}{2}}$ is

A. $\dfrac{1}{4}$

B. $\dfrac{1}{20}$

C. $\dfrac{1}{5}$

D. $\dfrac{1}{10}$

E. $\dfrac{3}{20}$

10. *[3 marks]*

The determinant of

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^T \begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 5 \\ 1 & 1 \end{pmatrix}$$

is

A. -24

B. $\dfrac{3}{2}$

C. 24

D. $-\dfrac{5}{2}$

E. $-\dfrac{3}{2}$

11. *[3 marks]*

To approximate $\displaystyle\int_0^1 \sqrt{1+x^3}\,dx$ dividing the interval $[0,1]$ into 4 equal parts and using the Trapezoidal Rule, $T_4 =$

A. $\dfrac{1}{4}\left[1+\sqrt{1+\dfrac{1}{64}}+\sqrt{1+\dfrac{1}{8}}+\sqrt{1+\dfrac{27}{64}}\right]$

B. $\dfrac{1}{4}\left[1+\sqrt{1+\dfrac{1}{64}}+\sqrt{1+\dfrac{1}{8}}+\sqrt{1+\dfrac{27}{64}}+\sqrt{2}\right]$

C. $\dfrac{1}{4}\left[1+\sqrt{1+\dfrac{1}{64}}+\sqrt{1+\dfrac{1}{8}}+2\sqrt{1+\dfrac{27}{64}}+\sqrt{2}\right]$

D. $\dfrac{1}{12}\left[1+2\sqrt{1+\dfrac{1}{64}}+4\sqrt{1+\dfrac{1}{8}}+2\sqrt{1+\dfrac{27}{64}}+\sqrt{2}\right]$

E. $\dfrac{1}{8}\left[1+2\sqrt{1+\dfrac{1}{64}}+2\sqrt{1+\dfrac{1}{8}}+2\sqrt{1+\dfrac{27}{64}}+\sqrt{2}\right]$

12. *[3 marks]*

If $\dfrac{dy}{dx} = xy$ and $y = 1$ when $x = 1$, then when $x = 0$, $y =$

A. 0

B. $\dfrac{1}{2}$

C. $\sqrt{e}$

D. $\dfrac{1}{\sqrt{e}}$

E. 2

13. *[3 marks]*

If $f(x, y) = 3x^3 - 2x^2 + 3y - y^3$ then the number of critical points of f is

A. 0

B. 1

C. 2

D. 3

E. 4

14. *[3 marks]*

The graph of $z = y^2 - x^3 + x$ has

A. one relative maximum and one critical point which is not an extremum

B. one relative minimum and one critical point which is not an extremum

C. one relative maximum and one relative minimum

D. two relative maxima (maximums for those who don't know Latin)

E. two critical points which are not extrema

15. *[3 marks]*

$$\int_{-1}^{1}\int_{x}^{x^2}(6x-12xy)\,dy\,dx =$$

A. -6

B. -4

C. 12

D. 4

E. 0

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[10 marks]*

An investor has a choice of two different investments.

- an ordinary annuity which costs $50,000 and has 16 semiannual payments at interest rate 4% compounded semiannually.
- 47 $1,000 bonds all of the same issue. Each bond matures in 8 years and has semiannual coupons worth $25 each. The total price of the 47 bonds is $50,000.

(a) *[4 marks]*

What is the amount of each payment of the annuity?

(b) *[6] marks*

Which investment has the higher yield?
Show your work.

B2. *[12 marks]*

A manufacturer has been selling 1000 bicycles a week at $450 each. A market survey indicates that for each $10 rebate offered to the buyer, the number of bicycles sold will increase by 100 per week.

[You do not have to do (a) to do (b) and (c).]

(a) *[3 marks]*
If p is the price of one bicycle, and q is the number of bicycles sold in a week, what is the demand function?

(b) *[5 marks]*

How large a rebate should the company offer the buyer to maximize its revenue?

(c) *[4 marks]*

If its weekly cost function is $C = 68,000 + 150q$, where C is in dollars, how large a rebate should the company offer to maximize profit?

B3. *[12 marks]*

(a) *[6 marks]*

Evaluate $\displaystyle\int_1^4 x^2 \ln(3x)\, dx$

(b) *[6 marks]*

Evaluate $\displaystyle\int_1^\infty \frac{3}{x^2(x+3)}\, dx$ or show that the integral diverges.

B4. *[11 marks]*

(a) *[6 marks]*

Given: $x^2 = y^2 + z^2 - 2xy$.

Find: $\dfrac{\partial^2 z}{\partial y \partial x}$ and show that it can be written in the form: $\dfrac{2xy}{z^3}$

(b) *[5 marks]*

If $w = x^2 + 2xy + 3y^2$, $x = e^r$, and $y = \ln(r+s)$ then find $\dfrac{\partial w}{\partial s}$ when $r = 0$ and $s = e$.

B5. *[10 marks]*

Use the method of Lagrange multipliers to find the minimum value of

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to the constraint

$$3x + 2y + z = 6$$

[You do **not** have to show that the value you find is in fact a minimum; but you **do** have to use Lagrange multipliers.]

Solutions to April 2004 Exam, MAT133Y

PART A

1. ANSWER: **D**.

$$\ln f(x) = 3\ln(x^2+e^{2x}) - 2x - \frac{2}{3}\ln(1+x-x^2)$$
$$\frac{1}{f(x)}f'(x) = \frac{3(2x+2e^{2x})}{x^2+e^{2x}} - 2 - \frac{2(1-2x)}{3(1+x-x^2)}$$
$$\frac{f'(0)}{f(0)} = \frac{3\cdot 2}{1} - 2 - \frac{2}{3} = \frac{10}{3}$$
$$f(0) = 1$$
$$f'(0) = \boxed{\frac{10}{3}}$$

2. ANSWER: **A**.

For f to be diff. at $x=0$, f must be cont. at $x=0$ so $e^0 = a\cdot 0 + b$ i.e. $b=1$ and $f'(0)$ must be the same from both sides, so

$$-e^0 = a, \quad \text{i.e.} \quad a=-1$$
$$f(x) = \begin{cases} e^{-x} & x \le 0 \\ -x+1 & x>0 \end{cases} = 0 \quad \text{only if} \quad \boxed{x=1}$$

3. ANSWER: **E**.

$$f'(x) = 2(1-\ln x)(-\frac{1}{x}) = \frac{2(\ln x - 1)}{x}$$
$$f''(x) = 2\Big[\frac{1-(\ln x - 1)}{x^2}\Big] = \frac{2(2-\ln x)}{x^2}$$

which changes sign at $\ln x = 2$, i.e. $\boxed{x=e^2}$

4. ANSWER: **A**.

$y' = 1+\ln x = 0$ when $\ln x = -1$, i.e. $x = \frac{1}{e}$

	y'	y
$(0, \frac{1}{e})$	-	dec
$(\frac{1}{e}, e)$	+	inc

but $x=\frac{1}{e}$ is the min.

$\lim_{x\to 0} x\ln x = 0$ and $f(e) = e\ln e = e$ so $x=e$ $\boxed{f(e)=e}$ is the max.

5. ANSWER: **B**.

$$y = x^{\frac{1}{1-x}} \qquad \ln y = \frac{\ln x}{1-x}$$

$$\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1}$$

because

$$\frac{0}{0} = -1$$

so

$$\lim_{x\to 1} y = e^{-1} = \boxed{\frac{1}{e}}$$

6. ANSWER: **A**.

$$F(x) = -\int_7^x e^t \ln|t|\,dt$$

$$F'(x) = -e^x \ln|x|$$

$$F''(x) = -e^x \ln|x| - \frac{e^x}{x} \quad \text{so}$$

$$F''(1) = \boxed{-e}$$

7. ANSWER: **A**.

$$\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(x^2+1)\Big|_0^1$$

$$= \frac{1}{2}\ln 2 = \boxed{\ln\sqrt{2}}$$

8. ANSWER: **A**.
$f(x) > 0$ always, so

$$A = \int_0^{\ln 5} \frac{e^x}{1+e^x}\,dx$$

$$= \ln(1+e^x)\Big|_0^5$$

$$= \ln(1+5) - \ln(1+1)$$

$$= \ln\left(\frac{6}{2}\right)$$

$$= \boxed{\ln 3}$$

9. ANSWER: **C**.

$x^{\frac{2}{3}} = x^{\frac{3}{2}}$ when $x = 0$ and $x = 1$ $\left[0 = x^{\frac{3}{2}} - x^{\frac{2}{3}} = x^{\frac{2}{3}}(x^{\frac{5}{6}} - 1)\right]$.

At $x = \dfrac{1}{2}$ $x^{\frac{2}{3}} = 4^{\frac{1}{3}}$ and $x^{\frac{3}{2}} = \dfrac{1}{\sqrt{8}}$ so $x^{\frac{2}{3}} > x^{\frac{3}{2}}$ $0 < x < 1$.

$$A = \int_0^1 (x^{\frac{2}{3}} - x^{\frac{3}{2}})\, dx = \left[\frac{3}{5}x^{\frac{5}{3}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^1 = \boxed{\frac{1}{5}}$$

10. ANSWER: **D**.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 6 & 8 \end{pmatrix}$$

whose $\det = -2$.

$$\det \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^T = -5$$

and

$$\det \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix}} = \frac{1}{-4}.$$

So

$$\det A = (-5)(-\frac{1}{4})(-2) = \boxed{-\frac{5}{2}}$$

11. ANSWER: **E**.

$$\frac{b-a}{n} = \frac{1}{4} = \triangle x \quad T_x = \frac{\triangle x}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right]$$

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad x_4 = 1,$$
$$y_0 = 1, \quad y_1 = \sqrt{1 + \frac{1}{64}}, \quad y_2 = \sqrt{1 + \frac{1}{8}}, \quad y_3 = \sqrt{1 + \frac{27}{64}}, \quad y_4 = \sqrt{2}$$

By inspection of the formula for T_4, E is the answer.

12. ANSWER: **D**.

$$\int \frac{dy}{y} = \int x\,dx \qquad \ln|y| = \frac{x^2}{2} + C$$

$$y = Ae^{\frac{x^2}{2}} \qquad 1 = Ae^{\frac{1}{2}}$$

so

$$A = e^{-\frac{1}{2}} \qquad y = \frac{e^{\frac{x}{2}}}{\sqrt{e}}$$

so

$$y(0) = \boxed{\frac{1}{\sqrt{e}}}$$

13. ANSWER: **E**.

$$\frac{\partial f}{\partial x} = 9x^2 - 4x = x(9x-4) = 0$$

$$\frac{\partial f}{\partial y} = 3 - 3y^2 = 3(1-y)(1+y) = 0$$

$$\text{crit pts : } \quad (0,1),\ (0,-1),\ (\frac{4}{9},1),\ (\frac{4}{9},-1): \quad \text{there are } \boxed{4}$$

14. ANSWER: **B**.

$$\frac{\partial z}{\partial x} = -3x^2 + 1$$

$$\frac{\partial z}{\partial y} = 2y = 0$$

$$\text{crit pts } (\frac{1}{\sqrt{3}},0),\ (-\frac{1}{\sqrt{3}},0)$$

$$z_{xx} = -6x \quad z_{yy} = 2 \quad z_{xy} = 0 \quad D = -12x$$

$$D(\frac{1}{\sqrt{3}},0) = -\frac{12}{\sqrt{3}} < 0 \quad \boxed{\text{not an extremum}}$$

$$D(-\frac{1}{\sqrt{3}},0) = \frac{12}{\sqrt{3}} > 0 \quad \boxed{\text{an extremum}}$$

$$\text{and } \ z_{yy} = 2 > 0 \ \text{ so } \boxed{\text{1 local min.}}$$

15. ANSWER: **B**.

$$\int_{-1}^{1}\int_{x}^{x^2} (6x - 12xy)\,dy\,dx$$

$$= \int_{-1}^{1} \Big[6xy - 6xy^2\Big]_{y=x}^{x^2} \,dx = \int_{-1}^{1} (6x^3 - 6x^5 - 6x^2 + 6x^3)\,dx$$

$$= \int_{-1}^{1} (12x^3 - 6x^5 - 6x^2)\,dx = -12\int_{0}^{1} x^2\,dx = \frac{-12}{3} = \boxed{-4}$$

PART B

B1.

(a)

$$50,000 = Ra_{\overline{16}|.02}$$

$$R = \frac{50,000}{a_{\overline{16}|.02}} = \frac{50,000 \times .02}{1-(1.02)^{-16}} = \boxed{\$3,682.51}$$

(b)

$$50,000 = 47,000(1+i)^{-16} + 47.25a_{\overline{16}|i}$$

Since the yield on the annuity is 2% per half-year, the question is how i compares to 2% .

$$\text{If} \quad i = .02, \quad P = 47,000(1.02)^{-16} + 47.25a_{\overline{16}|.02}$$

$$P = 50,190.76 > 50,000 \text{ the actual price}$$

At $i = .02$ the price comes out too high, so the yield is too low. The actual yield of the bond $> 2\%$.

$$\boxed{\text{The bond has the higher yield.}}$$

B2. Let $x =$ no. of \$10 rebates in the price.

$$p = 450 - 10x \qquad q = 1000 + 100x$$

(a)

$$x = \frac{450-p}{10} \qquad \frac{q-1000}{100} = x$$

$$\text{So} \quad \frac{450-p}{10} = \frac{q-1000}{100}$$

$$\boxed{p = 550 - \frac{q}{10} \qquad \text{or} \qquad q = 5500 - 10p}$$

(b)

$$revenue = R = pq = p(5500-10p)$$

$$\frac{dR}{dp} = 5500 - 20p = 0 \quad \text{when} \quad \boxed{p = \$275}$$

$$\text{which is a} \quad \boxed{\text{rebate of } 175}$$

Could also use

$$R = (450 - 10x)(1000 + 100x)$$
$$\frac{dR}{dx} = 35,000 - 2,000x = 0 \quad \text{when} \quad x = 17.5$$

and the rebate is again $10 \times 17.5 = \$175$.

In both ways of doing the problem the answer is a max. because R is an inverted parabola which has max. at the only point where $R' = 0$.

(c)

Profit $= \Pi = R - C = R(x) - \Big[68,000 + 150(1000 + 100x)\Big]$

$$\frac{dC}{dx} = \frac{dR}{dx} - 15000 = 20,000 - 2000x = 0 \qquad \text{when} \qquad x = 10$$

when rebate $\boxed{= \$100}$.

This is a max. for the same reason as in (b).

B3.

(a) Let

$$u = \ln 3x \qquad dv = x^2\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = \frac{x^3}{3}$$

$$\begin{aligned}\int_1^4 x^2 \ln 3x\,dx &= \frac{x^3}{3}\ln 3x\Big|_1^4 - \int_1^4 \frac{x^2}{3}\,dx \\ &= \frac{64}{3}\ln 12 - \frac{1}{3}\ln 3 - \frac{x^3}{9}\Big|_1^4 \\ &= \boxed{\frac{64}{3}\ln 12 - \frac{1}{3}\ln 3 - \left(\frac{64}{9} - \frac{1}{9}\right)} \\ &= \boxed{21\ln 3 + \frac{128}{3}\ln 2 - 7 \approx 45.6}\end{aligned}$$

(b)

$$\frac{3}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

$$Ax(x+3) + B(x+3) + Cx^2 = 3$$

$$x = 0 \quad \text{gives} \quad 3B = 3 \quad \text{or} \quad B = 1$$

$$x = -3 \quad \text{gives} \quad 9C = 3 \quad \text{or} \quad C = \frac{1}{3}$$

Since the coeff of x^2 is $A+C=0$, $A=-C=-\frac{1}{3}$

$$\begin{aligned}\int_1^\infty \frac{3}{x^2(x+3)}\,dx &= \lim_{R\to\infty}\int_1^R \Big[-\frac{1}{3}\cdot\frac{1}{x}+\frac{1}{3}\cdot\frac{1}{x+3}+\frac{1}{x^2}\,dx\Big]\\ &= \lim_{R\to\infty}\Big[-\frac{1}{3}\ln|x|+\frac{1}{3}\ln|x+3|-\frac{1}{x}\Big]_1^R\\ &= \lim_{R\to\infty}\Big[\frac{1}{3}\ln\Big|\frac{x+3}{x}\Big|-\frac{1}{x}\Big]_1^R\\ &= \lim_{R\to\infty}\Big[\big(\frac{1}{3}\ln\frac{R+3}{R}-\frac{1}{R}\big)-\big(\frac{1}{3}\ln 4-4\big)\Big]\\ &= \boxed{1-\frac{1}{3}\ln 4\approx .538}\end{aligned}$$

B4.

(a) Differentiating by x, considering z as a fcu. of x and y

$$2x=2z\frac{\partial z}{\partial x}-2y \qquad \Big(\frac{\partial z}{\partial x}=\frac{x+y}{z}\Big)$$

Now differentiating by y,

$$0=2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}+2z\frac{\partial^2 z}{\partial y\partial x}-2$$

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{1-\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}}{z} \qquad \text{we need } \frac{\partial z}{\partial y}.$$

$$0=2y+2z\frac{\partial z}{\partial y}-2x \quad \text{so} \quad \frac{\partial z}{\partial y}=\frac{x-y}{z}$$

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{1-\frac{(x-y)(x+y)}{z^2}}{z}=\frac{z^2-(x^2-y^2)}{z^3}$$

But the original equation says $z^2+y^2-x^2=2xy$, so $\frac{\partial^2 z}{\partial y\partial x}=\frac{2xy}{z^3}$

One can also take

$$\frac{\partial z}{\partial x}=\frac{x+y}{z}$$

so

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{z-(x+y)\frac{\partial z}{\partial y}}{y^2}=\frac{z-(x+y)\frac{(x-y)}{z}}{z^2}$$

if we find $\frac{\partial z}{\partial y}$ as above. This lead to the same place.

(b)

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x+2y)\cdot 0 + (2x+6y)\cdot\frac{1}{r+s}\end{aligned}$$

When $r=0$ and $s=e$, $x=1$ and $y=\ln(0+e)=1$ so

$$\frac{\partial w}{\partial s} = \frac{2+6}{0+e} = \boxed{\frac{8}{e}}$$

B5.

$$\begin{aligned}L &= x^2+y^2+z^2-\lambda(3x+2y+z-6) \\ L_x &= 2x-3\lambda=0 \qquad x=\frac{3\lambda}{2} \\ L_y &= 2y-2\lambda=0 \qquad y=\lambda \\ L_z &= 2z-\lambda=0 \qquad z=\frac{\lambda}{2}\end{aligned}$$

$$6=3x+2y+z=\frac{9\lambda}{2}+2\lambda+\frac{\lambda}{2}=7\lambda \quad \text{so} \quad \lambda=\frac{6}{7}$$

$$\boxed{x=\frac{3\lambda}{2}=\frac{9}{7}, \quad y=\lambda=\frac{6}{7}, \quad z=\frac{\lambda}{2}=\frac{3}{7}}$$

Point is $\boxed{\left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right)}$

and $f=x^2+y^2+z^2-\dfrac{81+36+9}{49}=\dfrac{126}{49}=\boxed{\dfrac{18}{7}}\approx 2.57$