FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2003

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. *[3 marks]*

A \$100,000 mortgage is to be repaid over 10 years by equal monthly payments made at the end of each month; that is, the first payment is one month after the loan is made. If interest is 10% compounded semiannually, the amount of each payment is

Ⓐ \$1,264.45

Ⓑ \$1,310.34

Ⓒ \$1,273.96

Ⓓ \$1,299.72

Ⓔ \$1,335.10

2. *[3 marks]*

A \$100 bond with 10 years until maturity has semiannual coupons at an annual coupon rate of 10%. If its annual yield is 9%, then its market price is

Ⓐ \$108.48

Ⓑ \$107.76

Ⓒ \$105.97

Ⓓ \$109.03

Ⓔ \$106.50

3. *[3 marks]*

If A is a 4×4 matrix and $\det(A) = 5$, then $\det(3A^2) =$

Ⓐ 225

Ⓑ 75

Ⓒ 2025

Ⓓ -75

Ⓔ 300

4. *[3 marks]*

$f(x) = (x^2 - 4)^{1/3}$ has

Ⓐ an absolute minimum at $x = 0$ and no absolute maximum

Ⓑ absolute minimums at $x = 2$ and $x = -2$ and a relative maximum at $x = 0$

Ⓒ a relative minimum at $x = 0$ and absolute maximums at $x = -2$ and $x = 2$

Ⓓ a relative maximum at $x = 0$ and no relative minimums

Ⓔ no relative maximums or minimums

5. *[3 marks]*

If the cost function is given by:

$$c = 0.01q^2 + 6q + 100$$

Then average cost $\bar{c}$ is minimized when $q =$

Ⓐ 300

Ⓑ 0

Ⓒ 1

Ⓓ 100

Ⓔ 1000

6. *[3 marks]*

$$\lim_{x \to +\infty} \frac{\ln(1 + e^{2x})}{x}$$

Ⓐ is 1

Ⓑ is 4

Ⓒ is 0

Ⓓ is 2

Ⓔ does not exist

7. *[3 marks]*

$$\int_e^{e^4} \frac{1}{x(\ln x)^{\frac{1}{2}}}\,dx =$$

Ⓐ $\dfrac{1}{e^2}$

Ⓑ $\dfrac{1}{2e^4}$

Ⓒ $\dfrac{1}{2e^4} - \dfrac{1}{2e}$

Ⓓ $2(e - \sqrt{e}\,)$

Ⓔ 2

8. *[3 marks]*

If $\displaystyle\int_1^x f(t)\,dt = e^x \ln x$ when $x > 0$, then $f(1) =$

Ⓐ e

Ⓑ 2

Ⓒ 1

Ⓓ 0

Ⓔ $e^x \ln x$

9. *[3 marks]*

$$\int \frac{x^2+1}{x^2+x}\,dx =$$

Ⓐ $x + \ln|x| - 2\ln|x+1| + C$

Ⓑ $\ln|x| + \ln|x+1| + C$

Ⓒ $x^{-1} + 2(x+1)^{-1} + C$

Ⓓ $x - \ln x + 2\ln(x+1) + C$

Ⓔ $\ln\left|\dfrac{x}{x+1}\right| + C$

10. *[3 marks]*

Using a subdivision of the interval $[1, 3]$ into 4 subintervals of equal length, the trapezoidal rule yields the following approximation for

$$\int_1^3 \frac{1}{\ln(x+1)}\,dx :$$

Ⓐ 1.8722

Ⓑ 2.7230

Ⓒ 1.7342

Ⓓ 0.9863

Ⓔ 1.9409

11. *[3 marks]*

If $a > 0$, $\displaystyle\int_0^\infty axe^{-ax}\,dx$

Ⓐ diverges

Ⓑ equals a

Ⓒ equals $\dfrac{1}{a}$

Ⓓ equals 1

Ⓔ equals e^{-a}

12. *[3 marks]*

If $\dfrac{dy}{dx} = 2xy$ and $y = e$ when $x = 0$ then, when $x = 1$, $y =$

Ⓐ 1

Ⓑ e

Ⓒ e^2

Ⓓ e^3

Ⓔ e^4

13. *[3 marks]*

If $f(x,y) = e^{xy}$, then when $x = 2$ and $y = 3$, $\dfrac{\partial^3 f}{\partial x \partial y^2} =$

Ⓐ $24e^6$

Ⓑ $16e^6$

Ⓒ $30e^6$

Ⓓ $12e^6$

Ⓔ $18e^6$

14. *[3 marks]*

Let $x(r,s)$ and y(r, s) be functions such that:

$$\begin{array}{lll} x(2,1) = 5 & \dfrac{\partial x}{\partial r}(2,1) = -7 & \dfrac{\partial x}{\partial s}(2,1) = -2 \\ y(2,1) = -3 & \dfrac{\partial y}{\partial r}(2,1) = 8 & \dfrac{\partial y}{\partial s}(2,1) = 4 \end{array}$$

If $z = 2x^2 + xy + 3y^2$, then when $(r,s) = (2,1)$, $\dfrac{\partial z}{\partial s} =$

Ⓐ 14

Ⓑ -223

Ⓒ 1

Ⓓ -86

Ⓔ -47

15. *[3 marks]*

For $p_A > 0$ and $p_B > 0$, products A and B have joint demand functions

$$q_A(p_A, p_B) = 10 - 4p_A - 2p_B - 3p_A^2 + p_B^2$$

and

$$q_B(p_A, p_B) = 7 + 4p_A - p_B - p_A^2 - 5p_B^2.$$

For which p_A and p_B are the two products complementary?

Ⓐ $p_A < 2$ and $p_B > 1$

Ⓑ $p_A < 1$ and $p_B > 2$

Ⓒ $p_A > 2$ and $p_B < 1$

Ⓓ $p_A < 2$ and $p_B < 1$

Ⓔ $p_A > 1$ and $p_B < 2$

PART B. WRITTEN-ANSWER QUESTIONS

B1.

(a) *[7 marks]*

Use the method of Lagrange multipliers to minimize $f(x, y) = x^2 y$ subject to the constraint $\frac{1}{x} + \frac{1}{y} = 1$.
[No need to justify that you are indeed at a minimum.]

(b) *[5] marks*

Given that x, y and z satisfy

$$xy + yz + z^3 x = 14$$

find $\frac{\partial z}{\partial x}$ when $(x, y, z) = (4, 2, 1)$

B2.

(a) *[7 marks]*
Find and classify all critical points of the function

$$f(x, y) = x^2 - 12y^2 + 4y^3 + 3y^4$$

(b) *[5 marks]*
Evaluate the following integral:

$$\int_0^1 \int_z^{z^2} \int_0^{2y} 30xyz \, dx \, dy \, dz$$

B3. Consider the function

$$f(x) = \begin{cases} x \ln|x| - x & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$$

(a) *[2 marks]*
Find all points where f is not continuous (if any), showing that your answer is correct. medskip

(b) *[9 marks]*
Sketch the graph of $y = f(x)$. A complete answer includes an explanation of all standard features of the graph.

(c) *[6 marks]*
Find the area bounded by the graph of $y = f(x)$ and the x axis, where

$$f(x) = \begin{cases} x \ln|x| - x & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$$

B4.

(a) *[8 marks]*

Mr. Abbott owes Mr. Costello two debts:

- $300 due in 5 years
- $100 plus interest at 7% compounded annually, due in 3 years.

They have agreed that the combined debt is to be settled with 3 payments:

- the first payment to be made now
- the second payment to be twice the amount of the first, to be made in 2 years
- the third payment to be twice the amount of the second, to be made in 4 years.

If money is worth 8% compounded quarterly, what is the amount (to the nearest cent) of the first payment?

(b) *[6 marks]*

Assume A is a square matrix such that

$$A \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 6 \\ 3 & 0 & 9 \end{bmatrix}.$$

Find A^{-1}.

Solutions to April 2003 Exam, MAT133Y

PART A

1. ANSWER: (B)

$$(1+i)^{12} = (1.05)^2$$
$$100,000 = Ra_{\overline{120}|i}$$

$$R = \frac{100,000i}{1-(1+i)^{-120}} = \frac{100,000[(1.05)^{\frac{1}{6}}-1]}{1-(1.05)^{-20}}$$

$$\boxed{R = \$1310.34}$$

2. ANSWER: (E)

$$P = 100(1.045)^{-20} + 5a_{\overline{20}|.045}$$

$$\boxed{P = \$106.50}$$

3. ANSWER: (C)

If A is $n \times n$, $\det(kA) = k^n \det A$

$$\det(3A^2) = 3^4 \det(A^2) = 3^4 \times 5^2 \quad \text{since} \quad \det(A^2) = (\det A)(\det A)$$

$$\boxed{\det(3A^2) = 2025}$$

4. ANSWER: (A)

$f'(x) = \frac{1}{3}(x^2-4)^{\frac{-2}{3}} \cdot 2x = \frac{2x}{3(x^2-4)^{\frac{2}{3}}}$ with critical points at $x = -2, 0, 2$;

but $f' < 0$ when $x < 0$

and $f' > 0$ when $x > 0$

so there is $\boxed{\text{an absolute minimum at } x = 0 \text{ and no other extrema of any kind.}}$

5. ANSWER: (D)

$\bar{c} = \frac{c}{q} = .01q + 6 + \frac{100}{q}$

$\frac{d\bar{c}}{dq} = .01 - \frac{100}{q^2} = 0$ when $q^2 = 10,000$

$$\boxed{q = 100}$$

and $\frac{d\bar{c}}{dq} = \frac{.01}{q^2}(q^2 - 10,000) \begin{cases} < 0 & 0 < q < 100 \\ > 0 & 100 < q \end{cases}$

6. ANSWER: (D)

$$\frac{\infty}{\infty} : \lim_{x\to\infty} \frac{\ln(1+e^{2x})}{x} = \lim_{x\to\infty} \frac{2e^{2x}}{1+e^{2x}}$$
$$\frac{\infty}{\infty} : \qquad = \lim_{x\to\infty} \frac{4e^{2x}}{2e^{2x}} = \boxed{2}$$

7. ANSWER: Ⓔ

Let $u = \ln x$; $du = \frac{dx}{x}$ and $u = 1$ when $x = e$, $u = 4$ when $x = e^4$.

$$\int_e^{e^4} \frac{1}{x(\ln x)^{\frac{1}{2}}}\,dx = \int_1^4 \frac{du}{u^{\frac{1}{2}}} = 2u^{\frac{1}{2}}\Big|_1^4$$
$$= 2(2-1) = \boxed{2}$$

8. ANSWER: Ⓐ

By the Fundamental Theorem of Calculus

$$f(x) = (e^x \ln x)' = e^x \ln x + \frac{e^x}{x}$$
$$f(1) = e \cdot 0 + \frac{e}{1} = \boxed{e}$$

9. ANSWER: Ⓐ

$$\frac{x^2+1}{x^2+x} = 1 + \frac{-x+1}{x(x+1)}$$
$$\frac{-x+1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \qquad A(x+1) + Bx = -x+1$$
$$x = 0 \Rightarrow A = 1$$
$$x = -1 \Rightarrow -B = 2 \Rightarrow B = -2$$
$$\int \frac{x^2+1}{x^2+x}dx = \int \left(1 + \frac{1}{x} - \frac{2}{x+1}\right)dx$$
$$= \boxed{x + \ln|x| - 2\ln|x+1| + C}$$

10. ANSWER: Ⓔ

$$T_4 = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4), \quad \Delta x = \frac{3-1}{4} = \frac{1}{2}$$
$$= \frac{1}{4}\left(\frac{1}{\ln 2} + \frac{2}{\ln 2.5} + \frac{2}{\ln 3} + \frac{2}{\ln 3.5} + \frac{1}{\ln 4}\right)$$
$$= \boxed{1.9409}$$

11. ANSWER: Ⓒ

$$\int_0^\infty axe^{-ax}\,dx = \lim_{R\to\infty}\int_0^R axe^{-ax}dx \qquad \text{Let } u = x, dv = ae^{-ax}dx$$
$$du = dx,\ v = -e^{-ax}$$

$$= \lim_{R\to\infty} \left[-xe^{-ax}\Big|_0^R + \int_0^R e^{-ax}\,dx \right]$$
$$= \lim_{R\to\infty} -Re^{-aR} - \lim_{R\to\infty} \frac{1}{a}e^{-ax}\Big|_0^R$$
$$= 0 - \lim_{R\to\infty} \frac{1}{a}(e^{-aR} - 1)$$
$$= \boxed{\tfrac{1}{a}}$$

12. ANSWER: Ⓒ

$$\frac{dy}{y} = 2x\,dx$$
$$\ln y = x^2 + C \qquad \ln e = C \text{ so } C = 1$$
$$\ln y = x^2 + 1 \quad \text{so when} \quad x = 1,\ \ln y = 2$$
$$y = \boxed{e^2}$$

13. ANSWER: Ⓑ

$$f_y = xe^{xy}$$
$$f_{yy} = x^2e^{xy}$$
$$f_{yyx} = 2xe^{xy} + yx^2e^{xy} = (2x + x^2y)e^{xy}$$
$$= (4 + 12)e^6 = \boxed{16e^6} \quad \text{at} \quad x = 2, y = 3.$$

14. ANSWER: Ⓓ

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (4x + y)\frac{\partial x}{\partial s} + (x + 6y)\frac{\partial y}{\partial s}$$
$$= (4 \cdot 5 - 3)(-2) + (5 - 6 \cdot 3)(4) \quad \text{when} \quad (r, s) = (2, 1)$$
$$= \boxed{-86}$$

15. ANSWER: Ⓒ

$$\frac{\partial q_A}{\partial p_B} = -2 + 2p_B = 2(p_B - 1); \qquad \frac{\partial q_B}{\partial p_A} = 4 - 2p_A = 2(2 - p_A)$$
$$\frac{\partial q_A}{\partial p_B} < 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} < 0 \quad \text{when} \quad \boxed{p_A > 2 \text{ and } p_B < 1}$$

PART B

B1.

(a)

$$\mathfrak{L} = x^2y - \lambda\left(\frac{1}{x} + \frac{1}{y} - 1\right)$$

$$\mathfrak{L}_x = 2xy + \frac{\lambda}{x^2} = 0 \qquad \mathfrak{L}_\lambda = -\left(\frac{1}{x} + \frac{1}{y} - 1\right) = 0$$

$$\mathfrak{L}_y = x^2 + \frac{\lambda}{y^2} = 0$$

$$x^2y^2 = -\lambda = 2x^3y$$

and because $\frac{1}{x} + \frac{1}{y} = 1, x \neq 0$ and $y \neq 0$. Dividing by x^2y, $y = 2x$.

$$\frac{1}{x} + \frac{1}{2x} = 1$$

$$\frac{3}{2x} = 1 \Rightarrow x = \frac{3}{2}$$

$$\frac{1}{x} + \frac{1}{y} = 1 \Rightarrow \frac{2}{3} + \frac{1}{y} = 1 \Rightarrow y = 3$$

so $\boxed{x = \frac{3}{2},\ y = 3}$, $\lambda = \frac{-81}{4}$.

(b)

$$y + y\frac{\partial z}{\partial x} + z^3 + 3z^2x\frac{\partial z}{\partial x} = 0$$

$$2 + 2\frac{\partial z}{\partial x} + 1 + 12\frac{\partial z}{\partial x} = 0.$$

$$\text{At} \quad (4,2,1) = (x,y,z) \qquad 3 + 14\frac{\partial z}{\partial x} = 0$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-3}{14}}$$

Alternatively $\frac{\partial z}{\partial x} = -\frac{y+z^3}{y+3z^2x} = \frac{-3}{14}$ at $(4,2,1)$.

B2.

(a)

$$f_x = 2x$$

$$f_y = -24y + 12y^2 + 12y^3 = 12y(y^2 + y - 2) = 12y(y+2)(y-1)$$

$$f_x = 0 \Rightarrow x = 0$$

$f_y = 0 \Rightarrow y = 0, 1, \text{ or } -2$

$$\boxed{\text{Crit pts: } (0,0),\ (0,1),\ (0,-2)}$$

$D = f_{xx}f_{yy} - f_{xy}^2 = 2(-24 + 24y + 36y^2) - 0^2$

$D = 24(3y^2 + 2y - 2)$

$$\boxed{\begin{array}{rll} D(0,0) & = -48 < 0 & \text{no extremum (or saddle pt)} \\ D(0,1) & = 72 > 0 & f_{xx} = 2 > 0 \text{ local min} \\ D(0,-2) & = 144 > 0 & f_{xx} = 2 > 0 \text{ local min} \end{array}}$$

$$\boxed{\begin{array}{rl} (0,0) & \text{Saddle} \\ \left.\begin{array}{r} (0,1) \\ (0,-2) \end{array}\right\} & \text{Local Mins} \end{array}}$$

(b)

$$\begin{aligned}
\int_0^1 \int_z^{z^2} \int_0^{2y} 30xyz\,dx\,dy\,dz &= \int_0^1 \int_z^{z^2} 15x^2 \Big|_{x=0}^{x=2y} yz\,dy\,dz \\
&= \int_0^1 \int_z^{z^2} 15 \cdot 4y^2 \cdot yz\,dy\,dz \\
&= \int_0^1 \int_z^{z^2} 60y^3 z\,dy\,dz \\
&= \int_0^1 15y^4 \Big|_{y=z}^{y=z^2} z\,dz \\
&= \int_0^1 15(z^8 - z^4) z\,dz \\
&= \int_0^1 15(z^9 - z^5)\,dz \\
&= 15\left(\frac{z^{10}}{10} - \frac{z^6}{6}\right)\Big|_0^1 \\
&= 15\left(\frac{1}{10} - \frac{1}{6}\right) = 15\left(\frac{-2}{30}\right) = \boxed{-1}
\end{aligned}$$

B3.

(a) The only difficulty is at $x = 0$:

$$\lim_{x\to 0} x\ln|x| = \lim_{x\to 0}\frac{\ln|x|}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -x = 0$$

So $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x\ln|x| - x) = 0 = f(0)$

f is continuous everywhere.

(b)

$f'(x) = \ln|x|$

crit pts at $x = \pm 1$ and 0

	f'	f
$(-\infty,-1)$	$+$	inc
$(-1,0)$	$-$	dec
$(0,1)$	$-$	dec
$(1,\infty)$	$+$	inc

$x = -1$ local max, $x = 1$ local min

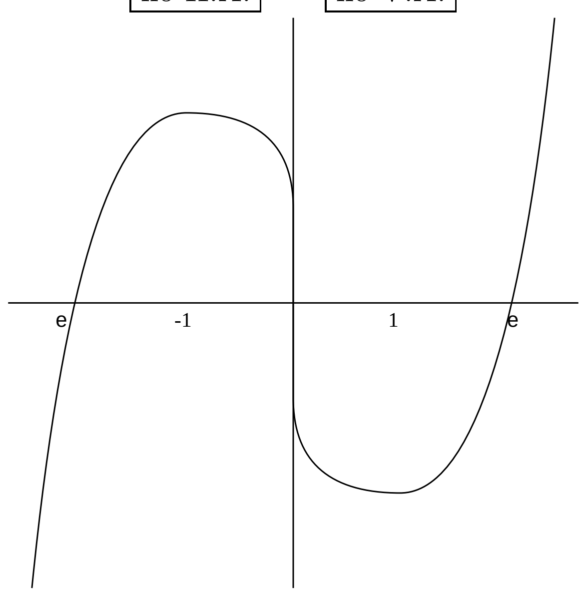
$f''(x) = \frac{1}{x}$

	f''	f
$(-\infty,0)$	$-$	conc down
$(0,\infty)$	$+$	conc up

$x = 0$ pt. of inflection

$$\lim_{x\to\infty} x\ln|x| - x = \lim_{x\to\infty} x\,[\ln|x| - 1] = \infty$$
$$\lim_{x\to-\infty} x\,[\ln|x| - 1] = -\infty$$

no H.A. **no V.A.**



B3. cont.

(c)

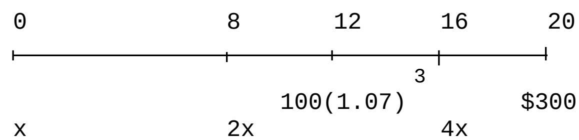
$$\text{Area} = \int_{-e}^{0} (x \ln|x| - x)dx - \int_{0}^{e} (x \ln|x| - x)dx$$
$$= -2 \int_{0}^{e} (x \ln|x| - x)dx$$

$$u = \ln|x| - 1 \quad dv = x\,dx$$
$$du = \tfrac{1}{x}dx \quad v = \tfrac{x^2}{2}$$

$$\text{Area} = -2 \left[\frac{x^2}{2}(\ln|x| - 1)\Big|_0^e - \frac{1}{2}\int_0^e x\,dx \right]$$
$$= 0 + \int_0^e x\,dx = \boxed{\frac{e^2}{2}}$$

Note that we have used $\lim_{x \to 0^+} x^2 \ln|x| = 0$ in evaluating the integral.

B4.



(a)

Let x be the first payment

$$x + 2x(1.02)^{-8} + 4x(1.02)^{-16} = 100(1.07)^3(1.02)^{-12} + 300(1.02)^{-20}$$

$$x = \frac{100(1.07)^3(1.02)^{-12} + 300(1.02)^{-20}}{1 + 2(1.02)^{-8} + 4(1.02)^{-16}}$$

$$\boxed{x = \$53.10}$$

(b) $\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix} = A^{-1} \begin{pmatrix} 0 & 1 & 6 \\ 3 & 0 & 9 \end{pmatrix}$ Let $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 6 \\ 3 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 3b & a & 6a+9b \\ 3d & c & 6c+9d \end{pmatrix}$$

Then $\begin{matrix} 3b = 2 & -1 = a \\ 3d = 1 & 0 = c \end{matrix}$ and $\begin{matrix} 6a + 9b = 0 \\ 6c + 9d = 3 \end{matrix}$

$$\begin{matrix} a = -1 & b = \frac{2}{3} \\ c = 0 & d = \frac{1}{3} \end{matrix}$$

$$\boxed{A^{-1} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}}$$